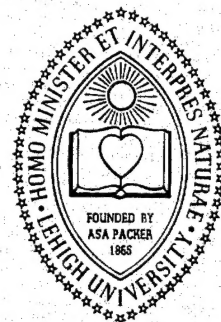


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LEHIGH UNIVERSITY



NORMAL AND RADIAL IMPACT OF COMPOSITES WITH EMBEDDED PENNY-SHAPED CRACKS

BY

G. C. SIH AND E. P. CHEN

FEBRUARY 1979

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16. Abstract A method is developed for the dynamic stress analysis of a layered composite containing an embedded penny-shaped crack and subjected to normal and radial impact. The material properties of the layers are chosen such that the crack lies in a layer of matrix material while the surrounding material possesses the average elastic properties of a two-phase medium consisting of a large number of fibers embedded in the matrix. Quantitatively, the time-dependent stresses near the crack border can be described by the dynamic stress intensity factors. Their magnitude depends on time, on the material properties of the composite and on the relative size of the crack compared to the composite local geometry. Results obtained show that, for the same material properties and geometry of the composite, the dynamic stress intensity factors for an embedded (penny-shaped) crack reach their peak values within a shorter period of time and with a lower magnitude than the corresponding dynamic stress intensity factors for a through-crack.					
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FOREWORD

This research work deals with the normal and radial impact of composites with embedded penny-shaped cracks which represents a portion of the program supported by the NASA-Lewis Research Center in Cleveland, Ohio. The program covers the period from February 13, 1978 to February 12, 1979 under Grant NSG 3179 and is conducted by the Institute of Fracture and Solid Mechanics at Lehigh University.

Professor George C. Sih served as the Principal Investigator while Dr. E. P. Chen was the Associate Investigator who is now employed by the Sandia Laboratory in New Mexico. The capable guidance of Dr. Christos C. Chamis who acted as the NASA Project Manager is very much appreciated. His encouragement has led to the success of this work.

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LIST OF SYMBOLS

a	- radius of crack
$A(s,p), B(s,p)$	- unknowns in dual integral equations
$A^{(i)}, B^{(i)}, C^{(i)}$	- coefficients for transform of solution, functions of (s,p)
b	- half of the thickness of the layer
Br	- Bromwich contour in the complex p -plane
c_{1j}, c_{2j}	- dilatational and shear wave speeds for medium j
$e^{(i)}$	- functions of (p,s) through γ_{ij}
$f^*(p)$	- Laplace transform of $f(t)$
$f^h(s)$	- Hankel transform of $f(x)$
$(f)_j$	- indicates that f is evaluated in medium j
$h(t)$	- Heaviside unit step function
$J_n(x)$	- Bessel function of order n
$k_1(t), k_2(t)$	- dynamic stress intensity factors
$M_I(\xi, n, p)$	- kernel of Fredholm integral equation
$M_{II}(\xi, n, p)$	
$P_I(s, p), P_{II}(s, p)$	- kernel in dual integral equations
r, θ, z	- cylindrical coordinates
r_1, θ_1	- crack tip polar coordinates
u_r, u_θ	- displacement components
t	- time
x, y, z	- rectangular coordinates - crack lies in the xy -plane
γ_{ij}	- exponents for transform of solution, functions of (p,s)
$\delta^{(i)}$	- functions of (p,s) through $e^{(i)}$
Δ_I, Δ_{II}	- functions of (p,s) through $\delta^{(i)}$
λ_1, λ_2	- Lamé coefficient

$\Lambda_I^*(\xi, p), \Lambda_{II}^*(\xi, p)$	- unknown in Fredholm integral equation
μ_1, μ_2	- shear modulus
ν_1, ν_2	- Poisson's ratio
ρ_1, ρ_2	- mass density
σ_0	- suddenly applied normal stress
$\sigma_r, \sigma_\theta, \sigma_z, \tau_{rz}$	- stress components
τ_0	- suddenly applied shear stress
ϕ_j, ψ_j	- scalar potentials for medium j
∇^2	- Laplacian operator

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ABSTRACT

A method is developed for the dynamic stress analysis of a layered composite containing an embedded penny-shaped crack and subjected to normal and radial impact. The material properties of the layers are chosen such that the crack lies in a layer of matrix material while the surrounding material possesses the average elastic properties of a two-phase medium consisting of a large number of fibers embedded in the matrix. Quantitatively, the time-dependent stresses near the crack border can be described by the dynamic stress intensity factors. Their magnitude depends on time, on the material properties of the composite and on the relative size of the crack compared to the composite local geometry. Results obtained show that, for the same material properties and geometry of the composite, the dynamic stress intensity factors for an embedded (penny-shaped) crack reach their peak values within a shorter period of time and with a lower magnitude than the corresponding dynamic stress factors for a through-crack.

^{*}This work was completed when Dr. Chen was a faculty member at Lehigh University.

INTRODUCTION

Advanced composite materials are multi-phased nonhomogeneous materials with anisotropic properties. This complicates the stress analysis for fracture, particularly if the loading is time-dependent and the geometry involves sharp edges such as a crack. As a result, conventional and mathematical techniques for dynamic fracture generally fail to yield accurate results.

An effective approach for finding dynamic stresses in a nonhomogeneous composite containing a through crack has been developed [1] by utilizing both the Laplace and Fourier transforms. The transient boundary, symmetry and continuity conditions were formulated by integral representations in terms of the rectangular Cartesian coordinates x and y and the results for the stress intensity factors are determined numerically by solving a standard integral equation in the Laplace transform plane. The crack geometry was assumed to be extended infinitely in the z -direction or through the side wall of the composite specimen. Many of the failures in composites, however, were observed [2] to initiate from embedded mechanical imperfections such as air bubbles, voids or cavities. Hence, a more realistic modeling of the actual flaw geometry would be an embedded crack that has finite dimensions in all directions. This immediately suggests a three-dimensional elastodynamic crack problem which cannot be solved effectively by analytical means unless symmetry prevails. One approach for obtaining a solution is to extend the integral transform formulation for a through crack in rectangular coordinates [1] to that of an embedded crack in cylindrical polar coordinates. This necessitates the use of Hankel transforms instead of Fourier transforms.

Although no attempt will be made to analyze the failure of the composite due to impact, the dynamic stress intensity factors $k_1(t)$ and $k_2(t)$ can be readily

used in a given fracture criterion, say the strain energy density theory [3], for determining the allowable level of impact load. The new results can also assist the construction of composite materials for establishing impact tolerance. In this case, failure is assumed to initiate from a damage zone of material in the composite that can be approximated by an embedded crack. The time-dependent characteristics of the stresses for the through and embedded crack geometries are compared and studied for different elastic properties and dimensions of the composite. In particular, the phenomenon of elastic waves reflecting from the crack to the interfaces within the composite can be exhibited numerically when their neighboring boundaries are sufficiently close to one another. As time becomes very large, all of the results in this report reduce to the corresponding static solutions [4].

AXIAL SYMMETRIC DEFORMATION: PENNY-SHAPED CRACK

Consider a penny-shaped crack of radius a that lies in a layer of material of thickness $2b$ with material properties μ_1, ν_1, ρ_1 . This layer is bonded between two media with properties μ_2, ν_2, ρ_2 as illustrated in Figure 1. With reference to the system of coordinates (x,y,z) , the z -axis coincides with the center of the crack and is normal to the crack situated in the xy -plane. The outer boundaries of the composite are assumed to be sufficiently far away from the crack such that the reflected waves will have a negligible influence on the local stresses. Only those impact loads that produce an axisymmetric wave pattern will be considered.

For an axially symmetric deformation field, material elements are displaced only in the radial and axial direction and remain unchanged in the θ -direction. With reference to the cylindrical polar coordinates (r,θ,z) in Figure 1, the

two nonzero displacement components can be expressed in terms of the wave potentials $\phi_j(r,z,t)$ and $\psi_j(r,z,t)$ as follows:

$$\begin{aligned}(u_r)_j &= \frac{\partial \phi_j}{\partial r} - \frac{\partial \psi_j}{\partial z} \\ (u_z)_j &= \frac{\partial \phi_j}{\partial z} + \frac{\partial \psi_j}{\partial r} - \frac{\psi_j}{r}\end{aligned}\tag{1}$$

where $j = 1$ refers to the layer with the crack and $j = 2$ to the surrounding material. The four nontrivial stress components are given by

$$\begin{aligned}(\sigma_r)_j &= 2\mu_j \frac{\partial}{\partial r} \left(\frac{\partial \phi_j}{\partial r} - \frac{\partial \psi_j}{\partial z} \right) + \lambda_j \nabla^2 \phi_j \\ (\sigma_\theta)_j &= 2\mu_j \frac{1}{r} \left(\frac{\partial \phi_j}{\partial r} - \frac{\partial \psi_j}{\partial z} \right) + \lambda_j \nabla^2 \phi_j \\ (\sigma_z)_j &= 2\mu_j \frac{\partial}{\partial z} \left(\frac{\partial \phi_j}{\partial z} + \frac{\partial \psi_j}{\partial r} + \frac{\psi_j}{r} \right) + \lambda_j \nabla^2 \phi_j \\ (\tau_{rz})_j &= \mu_j \left[\frac{\partial}{\partial z} \left(2 \frac{\partial \phi_j}{\partial r} - \frac{\partial \psi_j}{\partial z} \right) + \frac{\partial}{\partial r} \left(\frac{\partial \phi_j}{\partial r} + \frac{\psi_j}{r} \right) \right]\end{aligned}\tag{2}$$

in which λ_j and μ_j are the Lamé constants and ∇^2 represents the operator

$$\nabla^2 = \frac{\partial^2}{\partial r^2} + \frac{1}{r} \frac{\partial}{\partial r} + \frac{\partial^2}{\partial z^2}$$

The governing equations can thus be obtained from the equations of motion which yield

$$\frac{\partial^2 \phi_j}{\partial r^2} + \frac{1}{r} \frac{\partial \phi_j}{\partial r} + \frac{\partial^2 \phi_j}{\partial z^2} = \frac{1}{c_{1j}^2} \frac{\partial^2 \phi_j}{\partial t^2} \quad (3)$$

$$\frac{\partial^2 \psi_j}{\partial r^2} + \frac{1}{r} \frac{\partial \psi_j}{\partial r} - \frac{\psi_j}{r^2} + \frac{\partial^2 \psi_j}{\partial z^2} = \frac{1}{c_{2j}^2} \frac{\partial^2 \psi_j}{\partial t^2}$$

with c_{1j} and c_{2j} being the dilatational and shear wave speeds:

$$c_{1j} = \left(\frac{\lambda_j + 2\mu_j}{\rho_j} \right)^{1/2}, \quad c_{2j} = \left(\frac{\mu_j}{\rho_j} \right)^{1/2} \quad (4)$$

If the composite body is initially at rest, the Laplace transform of equations (3) further give

$$\frac{\partial^2 \phi_j^*}{\partial r^2} + \frac{1}{r} \frac{\partial \phi_j^*}{\partial r} + \frac{\partial^2 \phi_j^*}{\partial z^2} = \frac{p^2}{c_{1j}^2} \phi_j^* \quad (5)$$

$$\frac{\partial^2 \psi_j^*}{\partial r^2} + \frac{1}{r} \frac{\partial \psi_j^*}{\partial r} - \frac{\psi_j^*}{r^2} + \frac{\partial^2 \psi_j^*}{\partial z^2} = \frac{p^2}{c_{2j}^2} \psi_j^*$$

Here, p is the transform variable in the Laplace transform pair:

$$f^*(p) = \int_0^{\infty} f(t) \exp(-pt) dt \quad (6)$$

$$f(t) = \frac{1}{2\pi i} \int_{Br} f^*(p) \exp(pt) dp$$

The abbreviation Br stands for the Bromwich path of integration. Moreover, since the composite geometry is symmetrical about the xy -plane, it suffices to consider

only the solution in the upper half-space, $z \geq 0$. For the penny-shape crack geometry, the Hankel transform pair [5] may be used:

$$f^h(s) = \int_0^{\infty} x f(x) J_n(sx) dx$$

$$f(x) = \int_0^{\infty} s f^h(s) J_n(sx) ds$$
(7)

where J_n is the n th order Bessel function of the first kind. Applying equations (7) to (5), the following results are obtained:

$$\phi_1^*(r, z, p) = \int_0^{\infty} [A^{(1)}(s, p) e^{-\gamma_{11} z} + A^{(2)}(s, p) e^{\gamma_{11} z}] J_0(rs) ds$$

$$\psi_1^*(r, z, p) = \int_0^{\infty} [B^{(1)}(s, p) e^{-\gamma_{21} z} + B^{(2)}(s, p) e^{\gamma_{21} z}] J_1(rs) ds$$
(8)

for the cracked layer and

$$\phi_2^*(r, z, p) = \int_0^{\infty} C^{(1)}(s, p) e^{-\gamma_{12} z} J_0(rs) ds$$

$$\psi_2^*(r, z, p) = \int_0^{\infty} C^{(2)}(s, p) e^{-\gamma_{22} z} J_1(rs) ds$$
(9)

for the surrounding material. The quantities γ_{ij} are given by

$$\gamma_{1j} = (s^2 + \frac{p^2}{c_{1j}^2})^{1/2}, \quad \gamma_{2j} = (s^2 + \frac{p^2}{c_{2j}^2})^{1/2}$$
(10)

The six unknowns $A^{(1)}, A^{(2)}, \dots, C^{(2)}$ are determined from a given set of transient boundary, symmetry and continuity conditions.

NORMAL IMPACT

Let the penny-shaped crack be subjected to a uniform impact load* such that the upper and lower surface will move in the opposite direction. The magnitude of this normal load is σ_0 and since it is applied suddenly from $t = 0$ and maintained at a constant value thereafter, the Heaviside unit step function, $H(t)$, will be used, i.e., $-\sigma_0 H(t)$. Making use of equations (6), the conditions on the plane $z = 0$ for $r \leq a$ and $r \geq a$ take the forms

$$(\sigma_z^*)_1(r, 0, p) = -\frac{\sigma_0}{p}; (\tau_{rz}^*)_1(r, 0, p) = 0, 0 \leq r < a \quad (11)$$

$$(u_z^*)_1(r, 0, p) = 0; (\tau_{rz}^*)_1(r, 0, p) = 0, r \geq a$$

If the interfaces at $z = \pm b$ is bonded perfectly, the stresses and displacements can then be considered continuous across these planes, i.e.,

$$(\sigma_z^*)_1(r, b, p) = (\sigma_z^*)_2(r, b, p) \quad (12)$$

$$(\tau_{rz}^*)_1(r, b, p) = (\tau_{rz}^*)_2(r, b, p)$$

*There is no loss in generality in formulating the problem in terms of a uniform step load. The principle of superposition may be used to obtain the solution for general loading from a series of step loading solutions as discussed in [1].

and

$$(u_r^*)_1(r, b, p) = (u_r^*)_2(r, b, p) \quad (13)$$

$$(u_z^*)_1(r, b, p) = (u_z^*)_2(r, b, p)$$

Under these considerations, the six functions $A^{(1)}, A^{(2)}, \dots, C^{(2)}$ may be expressed in terms of a single unknown $A(s, p)$ as indicated by equations (A.1) in the Appendix.

Fredholm integral equations. Without going into details, the function $A(s, p)$ can be obtained from the system of dual integral equations

$$\int_0^\infty A(s, p) J_0(rs) ds = 0, \quad r \geq a \quad (14)$$

$$\int_0^\infty s P_I(s, p) A(s, p) J_0(rs) ds = - \frac{\sigma_0}{2\mu_1(1-\kappa_1^2)p}, \quad r < a$$

in which $P_I(s, p)$ is a known function:

$$\begin{aligned} P_I(s, p) = & \frac{1}{s\Delta_I(1-\kappa_1^2)} \left\{ \left[\frac{1}{4} (s^2 + \gamma_{21}^2)^2 - s^2 \gamma_{11} \gamma_{21} \right] [\delta^{(2)} - \delta^{(3)} e^{-2(\gamma_{11} + \gamma_{21})b}] \right. \\ & + s(s^2 + \gamma_{21}^2) e^{-(\gamma_{11} + \gamma_{21})b} [\gamma_{21}(\delta^{(1)} \delta^{(4)} - \delta^{(2)} \delta^{(3)}) - \gamma_{11}] \\ & \left. + \left[\frac{1}{4} (s^2 + \gamma_{21}^2)^2 + s^2 \gamma_{11} \gamma_{21} \right] [\delta^{(4)} e^{-2\gamma_{21}b} - \delta^{(1)} e^{-2\gamma_{11}b}] \right\} \quad (15) \end{aligned}$$

The form of $A(s,p)$ that satisfies equations (14) can be found from Copson [6]:

$$A(s,p) = - \sqrt{\frac{2s}{\pi}} \frac{\sigma_0 a^{5/2}}{2\mu_1 p(1-\kappa_1^2)} \int_0^1 \sqrt{\xi} \Lambda_I^*(\xi,p) J_{1/2}(sa\xi) d\xi \quad (16)$$

Here, $J_{1/2}$ is the half order Bessel function of the first kind and $\Lambda_I^*(\xi,p)$ satisfies the Fredholm integral equation

$$\Lambda_I^*(\xi,p) + \int_0^1 \Lambda_I^*(\eta,p) M_I(\xi,\eta,p) d\eta = \xi \quad (17)$$

whose kernel

$$\begin{aligned} M_I(\xi,\eta,p) &= \sqrt{\xi\eta} \int_0^\infty s [P_I(\frac{s}{a},p) - 1] J_{1/2}(s\xi) J_{1/2}(s\eta) ds \\ &= \frac{2}{\pi} \int_0^\infty [P_I(\frac{s}{a},p) - 1] \sin(s\xi) \sin(s\eta) ds \end{aligned} \quad (18)$$

is symmetric in ξ and η . Figures 2 to 4 show the numerical results of equation (17) by varying μ_2/μ_1 and a/b while $\rho_1 = \rho_2$ and $\nu_1 = \nu_2 = 0.29$ are kept the same for all cases. The function $\Lambda_I^*(\xi,p)$ evaluated at the crack border, $\xi = 1$, governs the contribution of the geometric and material parameters on $k_1^*(p)$ which represents the Laplace transform of the stress intensity factor.

Stress intensity factor for normal impact. In order to evaluate $k_1^*(p)$ or $k_1(t)$, the stresses in the matrix layer are first expanded in terms of the local coordinates r_1 and θ_1 for small values of r_1 . The local coordinates (r_1, θ_1) are related to (r, θ) in Figure 1 as follows:

$$a + r_1 \cos \theta_1 = r \cos \theta \quad (19)$$

$$r_1 \sin \theta_1 = r \sin \theta$$

The leading term in the Laplace transform of the local stresses that possess the $1/\sqrt{r_1}$ singularity is

$$k_1^*(p) = \frac{\Lambda_I^*(1,p)}{p} \frac{2}{\pi} \sigma_0 \sqrt{a} \quad (20)$$

Application of the Laplace inversion theorem yields the dynamic stress field around the crack border as a function of time. The result is

$$\begin{aligned} (\sigma_r)_1(r_1, \theta_1, t) &= \frac{k_1(t)}{\sqrt{2r_1}} \cos \frac{\theta_1}{2} \left(1 - \sin \frac{\theta_1}{2} \sin \frac{3\theta_1}{2}\right) + O(r_1^0) \\ (\sigma_\theta)_1(r_1, \theta_1, t) &= \frac{k_1(t)}{\sqrt{2r_1}} 2\nu_1 \cos \frac{\theta_1}{2} + O(r_1^0) \\ (\sigma_z)_1(r_1, \theta_1, t) &= \frac{k_1(t)}{\sqrt{2r_1}} \cos \frac{\theta_1}{2} \left(1 + \sin \frac{\theta_1}{2} \sin \frac{3\theta_1}{2}\right) + O(r_1^0) \\ (\tau_{rz})_1(r_1, \theta_1, t) &= \frac{k_1(t)}{\sqrt{2r_1}} \cos \frac{\theta_1}{2} \sin \frac{\theta_1}{2} \cos \frac{3\theta_1}{2} + O(r_1^0) \end{aligned} \quad (21)$$

and $k_1(t)$ becomes

$$k_1(t) = \frac{2\sigma_0\sqrt{a}}{\pi} \frac{1}{2\pi i} \int_{Br} \frac{\Lambda_I^*(1,p)}{p} e^{pt} dp \quad (22)$$

Note that equation (20) is, in fact, the Laplace transform of equation (22).

Hence, the functional dependence of r_1 and θ_1 is not affected by the Laplace

transformation and can be evaluated separately. This observation was first made by Sih, Ravera and Embley [7].

Making use of the results for $\Lambda_I^*(1,p)$ in Figures 2 to 4, $k_1(t)$ in equation (22) can be found as given in Figures 5 to 7. The dynamic stress intensity factors $k_1(t)$ for the penny-shaped crack exhibit an oscillatory behavior rising quickly to a peak. As time increases, all curves approach the static value of $k_1 = 2\sigma_0\sqrt{a}/\pi$ [4]. For a crack diameter to layer thickness ratio of $a/b = 1$, the peaks of the $k_1(t)$ curve are sensitive to changes in the shear moduli ratio μ_2/μ_1 . Figure 5 indicates that $k_1(t)$ tends to decrease in amplitude as μ_2/μ_1 is reduced from 0.1 to 10.0. The influence of the composite interface on $k_1(t)$ is exhibited in Figures 6 to 7. When the shear modulus of the surrounding material μ_2 is much smaller than the matrix layer with μ_1 , the dynamic crack border stress intensity increases as the crack diameter becomes large in comparison with the layer thickness. This effect is clearly evidenced in Figure 6. As expected, $k_1(t)$ increases with decreasing a/b when the shear modulus of the cracked layer is made smaller than the surrounding material, i.e., $\mu_1 < \mu_2$ as illustrated in Figure 7. The result of Embley and Sih [8] is recovered for the homogeneous case, $\mu_1 = \mu_2$.

RADIAL IMPACT

If the penny-shaped crack is sheared uniformly in the radial direction such that axial symmetry is preserved, then $\phi_j^*(r,z,p)$ and $\psi_j^*(r,z,p)$ in equations (8) and (9) remain valid. Let this shear of magnitude τ_0 be applied suddenly and hence the surface tractions, $-\tau_0 H(t)$, are to be specified for $0 \leq r < a$ with $H(t)$ being the Heaviside unit step function. Laplace transform of the conditions on the plane $z = 0$ thus become

$$(\tau_{rz}^*)_1(r,0,p) = -\frac{\tau_0}{p}; (\sigma_z^*)_1(r,0,p) = 0, 0 \leq r < a \quad (23)$$

$$(u_r^*)_1(r,0,p) = 0; (\sigma_z^*)_1(r,0,p) = 0, r \geq a$$

Continuity of the stresses across the interface $z = b$ is satisfied if

$$(\sigma_z^*)_1(r,b,p) = (\sigma_z^*)_2(r,b,p) \quad (24)$$

$$(\sigma_{rz}^*)_1(r,b,p) = (\sigma_{rz}^*)_2(r,b,p)$$

and the same requirement is imposed on the displacements:

$$(u_r^*)_1(r,b,p) = (u_r^*)_2(r,b,p) \quad (25)$$

$$(u_z^*)_1(r,b,p) = (u_z^*)_2(r,b,p)$$

Integral equations. As in the case of normal impact, the six unknown functions $A^{(1)}(s,p)$, $A^{(2)}(s,p), \dots, C^{(2)}(s,p)$ in equations (8) and (9) can be expressed in terms of a single unknown $B(s,p)$. Refer to equations (A.5) in the Appendix. Hence, equations (24) and (25) are satisfied. The remaining boundary conditions in equations (23) are employed to obtain the system of dual integral equations

$$\int_0^{\infty} B(s,p) J_1(rs) ds = 0, \quad r \geq a \quad (26)$$

$$\int_0^{\infty} s P_{II}(s,p) B(s,p) J_1(rs) ds = - \frac{\tau_0}{2\mu_1(1-\kappa_1^2)p}, \quad r < a$$

in which

$$P_{II}(s,p) = \frac{\Delta_I}{\Delta_{II}} P_I(s,p) \quad (27)$$

where $P_I(s,p)$ is already known through equation (15) while $\Delta_I(s,p)$ and $\Delta_{II}(s,p)$ are given by equations (A.2) and (A.6), respectively.

Solving for $B(s,p)$ [6], it can be shown that

$$B(s,p) = - \sqrt{\frac{\pi s}{2}} \frac{\tau_0 a^{5/2}}{4\mu_1 p(1-\kappa_1^2)} \int_0^1 \sqrt{\xi} \Lambda_{II}^*(\xi,p) J_{3/2}(sa\xi) d\xi \quad (28)$$

and $\Lambda_{II}^*(\xi,p)$ satisfies the Fredholm integral equation of the second kind:

$$\Lambda_{II}^*(\xi,p) + \int_0^1 \Lambda_{II}^*(\eta,p) M_{II}(\xi,\eta,p) d\eta = \xi \quad (29)$$

whose kernel takes the form

$$M_{II}(\xi,\eta,p) = \sqrt{\xi\eta} \int_0^{\infty} s [P_{II}(\frac{s}{a}, p) - 1] J_{3/2}(s\xi) J_{3/2}(s\eta) ds \quad (30)$$

Plots of $\Lambda_{II}^*(1,p)$ as a function of c_{21}/pa are shown in Figures 8 to 10 for different values of μ_2/μ_1 and a/h . The curves show that $\Lambda_{II}^*(1,p)$ rises rapidly at first and then levels off.

Stress intensity factor for radial impact. The dynamic crack border stress field corresponding to radial shear can be obtained in the same way and expressed in terms of the coordinates (r_1, θ_1) in equations (19):

$$\begin{aligned}
 (\sigma_r)_1(r_1, \theta_1, t) &= \frac{k_2(t)}{\sqrt{2r_1}} \sin \frac{\theta_1}{2} \left(2 + \cos \frac{\theta_1}{2} \cos \frac{3\theta_1}{2} \right) + O(r_1^0) \\
 (\sigma_\theta)_1(r_1, \theta_1, t) &= \frac{k_2(t)}{\sqrt{2r_1}} 2\nu_1 \sin \frac{\theta_1}{2} + O(r_1^0) \\
 (\sigma_z)_1(r_1, \theta_1, t) &= - \frac{k_2(t)}{\sqrt{2r_1}} \sin \frac{\theta_1}{2} \cos \frac{\theta_1}{2} \cos \frac{3\theta_1}{2} + O(r_1^0) \\
 (\tau_{rz})_1(r_1, \theta_1, t) &= \frac{k_2(t)}{\sqrt{2r_1}} \cos \frac{\theta_1}{2} \left(1 - \sin \frac{\theta_1}{2} \sin \frac{3\theta_1}{2} \right) + O(r_1^0)
 \end{aligned} \tag{31}$$

Note that $k_2(t)$ can be evaluated from

$$k_2(t) = \frac{\tau_0 \sqrt{a}}{4\pi i} \int_{Br} \frac{\Lambda_{II}^*(1, p)}{p} e^{pt} dp \tag{32}$$

once $\Lambda_{II}^*(1, p)$ as given by Figures 8 to 10 is known.

The numerical results in Figures 11 to 13 for $k_2(t)$ as a function of time refer to $\rho_1 = \rho_2$ and $\nu_1 = \nu_2 = 0.29$. The curve with $\mu_1 = \mu_2$ is the solution for the homogeneous material treated previously by Embley and Sih [8]. In general, $k_2(t)$ oscillates with time and can be greater or smaller than the corresponding homogeneous solution depending on whether $\mu_2/\mu_1 < 1$ or $\mu_2/\mu_1 > 1$. Figure 11 displays the variations of $k_2(t)$ for different values of μ_2/μ_1 while a/b is fixed at unity. The influence of the ratio of crack size with layer thickness

is exhibited in Figures 12 and 13 for $\mu_2/\mu_1 = 0.1$ and $\mu_2/\mu_1 = 10.0$, respectively. These two cases show the opposite effect which is to be expected.

CONCLUDING REMARKS

The previous discussion has shown that the dynamic stress intensity factors for an embedded crack can be evaluated analytically by a method similar to that developed for a through crack [1]. An important consideration is to compare the results for these two crack configurations and to draw some general conclusions. First of all, the $k_1(t)$ or $k_2(t)$ factor for the penny-shaped crack tends to rise more quickly than the through crack, i.e., the peak value of $k_1(t)$ or $k_2(t)$ is reached within a shorter period of time. This is because waves emanating from the neighboring points on the periphery of the penny-shaped crack interfere with each other much earlier as compared to a line (or plane) crack where the waves must travel from one end to the other before interference can take place. In general, the maximum value of $k_1(t)$ or $k_2(t)$ for an embedded crack is lower than that for a through crack. For example, Figure 5 gives a peak value of approximately 1.6 for $\pi k_1(t)/2\sigma_0\sqrt{a}$ which corresponds to $a/b = 1.0$ and $\mu_2/\mu_1 = 0.1$. This occurs at $c_{21}t/a \approx 1.6$ and yields $k_1(t) \approx 1.02 \sigma_0\sqrt{a}$. The corresponding case of a through crack [1] renders $k_1(t) \approx 2.40 \sigma_0\sqrt{a}$ and $c_{21}t/a \approx 3.0$. The difference in $k_1(t)$ is more than a factor of two and is more pronounced as the ratio a/b is increased. For embedded cracks that are non-circular in shape, approximate estimates of $k_1(t)$ can be made by taking the solution for the through crack as an upper limit and that of the circular crack as a lower limit.

In the absence of axisymmetry, the dynamic stress analysis will become exceedingly difficult and it will be more feasible to solve the crack problem numerically. In such cases, the solutions obtained here can be used to guide the development of numerical procedures.

APPENDIX: EXPRESSIONS FOR $A^{(i)}(s,p), \dots, C^{(i)}(s,p)$

Normal impact. The functions $A^{(1)}(s,p), A^{(2)}(s,p), \dots, C^{(2)}(s,p)$ for the wave potentials in equations (8) and (9) can be expressed in terms of a single unknown $A(s,p)$ for normal impact

$$A^{(1)}(s,p) = \left[\frac{1}{2} (s^2 + \gamma_{21}^2) (\delta^{(2)} + \delta^{(4)} e^{-2\gamma_{21}b}) - s\gamma_{11} e^{-(\gamma_{11} + \gamma_{21})b} \right] \frac{A(s,p)}{\Delta_I}$$

$$A^{(2)}(s,p) = - \left[s\gamma_{11} e^{-(\gamma_{11} + \gamma_{21})b} + \frac{1}{2} (s^2 + \gamma_{21}^2) e^{-2\gamma_{11}b} (\delta^{(1)} + \delta^{(3)} e^{-2\gamma_{21}b}) \right] \times \frac{A(s,p)}{\Delta_I}$$

$$B^{(1)}(s,p) = - \left[\delta^{(1)} A^{(1)} e^{-\gamma_{11}b} + \delta^{(2)} A^{(2)} e^{\gamma_{11}b} \right]$$

$$B^{(2)}(s,p) = - \left[\delta^{(3)} A^{(1)} e^{-\gamma_{11}b} + \delta^{(4)} A^{(2)} e^{\gamma_{11}b} \right] \quad (A.1)$$

$$C^{(1)}(s,p) = \frac{e^{\gamma_{12}b}}{s^2 - \gamma_{12}\gamma_{22}} \left[(s^2 - \gamma_{11}\gamma_{22}) A^{(1)} e^{-\gamma_{11}b} + (s^2 + \gamma_{11}\gamma_{22}) A^{(2)} e^{\gamma_{11}b} \right. \\ \left. - s(\gamma_{21} - \gamma_{22}) B^{(1)} e^{-\gamma_{21}b} + s(\gamma_{21} + \gamma_{22}) B^{(2)} e^{\gamma_{21}b} \right]$$

$$C^{(2)}(s,p) = \frac{e^{\gamma_{22}b}}{s^2 - \gamma_{12}\gamma_{22}} \left[s(\gamma_{12} - \gamma_{11}) A^{(1)} e^{-\gamma_{11}b} + s(\gamma_{11} + \gamma_{12}) e^{\gamma_{11}b} \right. \\ \left. + (s^2 - \gamma_{21}\gamma_{12}) B^{(1)} e^{-\gamma_{21}b} + (s^2 + \gamma_{21}\gamma_{12}) B^{(2)} e^{\gamma_{21}b} \right]$$

in which Δ_I stands for

$$\Delta_I(s,p) = \frac{p^2}{2c_{21}^2} \gamma_{11} [\delta^{(2)} + \delta^{(3)} e^{-2(\gamma_{11} + \gamma_{21})b} + \delta^{(4)} e^{-2\gamma_{21}b} + \delta^{(1)} e^{-2\gamma_{11}b}] \quad (A.2)$$

and $\delta^{(1)}, \delta^{(2)}, \dots, \delta^{(4)}$ are further expressed in terms of $e^{(1)}, e^{(2)}, \dots, e^{(8)}$ as the following:

$$\begin{aligned} \delta^{(1)}(s,p) &= (e^{(1)}e^{(6)} - e^{(2)}e^{(7)}) / (e^{(1)}e^{(6)} - e^{(2)}e^{(5)}) \\ \delta^{(2)}(s,p) &= (e^{(4)}e^{(6)} - e^{(2)}e^{(8)}) / (e^{(1)}e^{(6)} - e^{(2)}e^{(5)}) \\ \delta^{(3)}(s,p) &= (e^{(1)}e^{(7)} - e^{(3)}e^{(5)}) / (e^{(1)}e^{(6)} - e^{(2)}e^{(5)}) \\ \delta^{(4)}(s,p) &= (e^{(1)}e^{(8)} - e^{(4)}e^{(5)}) / (e^{(1)}e^{(6)} - e^{(2)}e^{(5)}) \end{aligned} \quad (A.3)$$

The quantities in equations (A.3) are complicated functions of the materials parameters and transform variables. They are given by

$$\begin{aligned} e^{(1)}(s,p) &= -s\gamma_{21} + \frac{s\mu_2}{\mu_1(s^2 - \gamma_{12}\gamma_{22})} \left[\frac{1}{2} (\gamma_{21} - \gamma_{22})(s^2 + \gamma_{22}^2) + \gamma_{22}(s^2 - \gamma_{21}\gamma_{12}) \right] \\ e^{(2)}(s,p) &= s\gamma_{21} - \frac{s\mu_2}{\mu_1(s^2 - \gamma_{12}\gamma_{22})} \left[\frac{1}{2} (\gamma_{21} + \gamma_{22})(s^2 + \gamma_{22}^2) - \gamma_{22}(s^2 + \gamma_{21}\gamma_{12}) \right] \\ e^{(3)}(s,p) &= \frac{1}{2} (s^2 + \gamma_{21}^2) - \frac{\mu_2}{\mu_1(s^2 - \gamma_{12}\gamma_{22})} \left[\frac{1}{2} (s^2 + \gamma_{22}^2)(s^2 - \gamma_{11}\gamma_{22}) \right. \\ &\quad \left. + s^2\gamma_{22}(\gamma_{11} - \gamma_{12}) \right] \end{aligned}$$

$$e^{(4)}(s,p) = \frac{1}{2} (s^2 + \gamma_{21}^2) - \frac{\mu_2}{\mu_1 (s^2 - \gamma_{12} \gamma_{22})} \left[\frac{1}{2} (s^2 + \gamma_{22}^2) (s^2 + \gamma_{11} \gamma_{22}) \right. \\ \left. - s^2 \gamma_{22} (\gamma_{11} + \gamma_{12}) \right]$$

$$e^{(5)}(s,p) = -\frac{1}{2} (s^2 + \gamma_{21}^2) + \frac{\mu_2}{\mu_1 (s^2 - \gamma_{12} \gamma_{22})} \left[s^2 \gamma_{12} (\gamma_{21} - \gamma_{22}) \right. \\ \left. + \frac{1}{2} (s^2 + \gamma_{22}^2) (s^2 - \gamma_{21} \gamma_{12}) \right] \quad (A.4)$$

$$e^{(6)}(s,p) = -\frac{1}{2} (s^2 + \gamma_{21}^2) - \frac{\mu_2}{\mu_1 (s^2 - \gamma_{12} \gamma_{22})} \left[s^2 \gamma_{12} (\gamma_{21} + \gamma_{22}) \right. \\ \left. - \frac{1}{2} (s^2 + \gamma_{22}^2) (s^2 + \gamma_{21} \gamma_{12}) \right]$$

$$e^{(7)}(s,p) = s \gamma_{11} - \frac{s \mu_2}{\mu_1 (s^2 - \gamma_{12} \gamma_{22})} \left[\gamma_{12} (s^2 - \gamma_{11} \gamma_{22}) + \frac{1}{2} (s^2 + \gamma_{22}^2) (\gamma_{11} - \gamma_{12}) \right]$$

$$e^{(8)}(s,p) = -s \gamma_{11} - \frac{s \mu_2}{\mu_1 (s^2 - \gamma_{12} \gamma_{22})} \left[\gamma_{12} (s^2 + \gamma_{11} \gamma_{22}) - \frac{1}{2} (s^2 + \gamma_{22}^2) (\gamma_{11} + \gamma_{12}) \right]$$

Radial impact. For radial impact, $A^{(1)}(s,p)$, $A^{(2)}(s,p)$, ..., $C^{(2)}(s,p)$ in equations (8) and (9) can be expressed in terms of $B(s,p)$ as

$$A^{(1)}(s,p) = - \left[s \gamma_{21} (\delta^{(2)} - \delta^{(4)} e^{-2\gamma_{21}b}) + \frac{1}{2} (s^2 + \gamma_{21}^2) e^{-(\gamma_{11} + \gamma_{21})b} \right] \frac{B(s,p)}{\Delta_{II}} \quad (A.5)$$

$$A^{(2)}(s,p) = \left[s \gamma_{21} e^{-2\gamma_{11}b} (\delta^{(1)} - \delta^{(3)} e^{-2\gamma_{21}b}) + \frac{1}{2} (s^2 + \gamma_{21}^2) e^{-(\gamma_{11} + \gamma_{21})b} \right] \\ \times \frac{B(s,p)}{\Delta_{II}}$$

where

$$\Delta_{II} = \frac{p^2}{2c_{21}^2} \gamma_{21} [\delta^{(2)} + \delta^{(3)} e^{-2(\gamma_{11} + \gamma_{21})b} - \delta^{(4)} e^{-2\gamma_{21}b} - \delta^{(1)} e^{-2\gamma_{11}b}] \quad (A.6)$$

The remaining functions $B^{(1)}(s,p)$, $B^{(2)}(s,p)$, etc., can be related to $B(s,p)$ through $A^{(1)}(s,p)$ and $A^{(2)}(s,p)$ since the last four expressions in equations (A.1) for normal impact also apply to radial impact.

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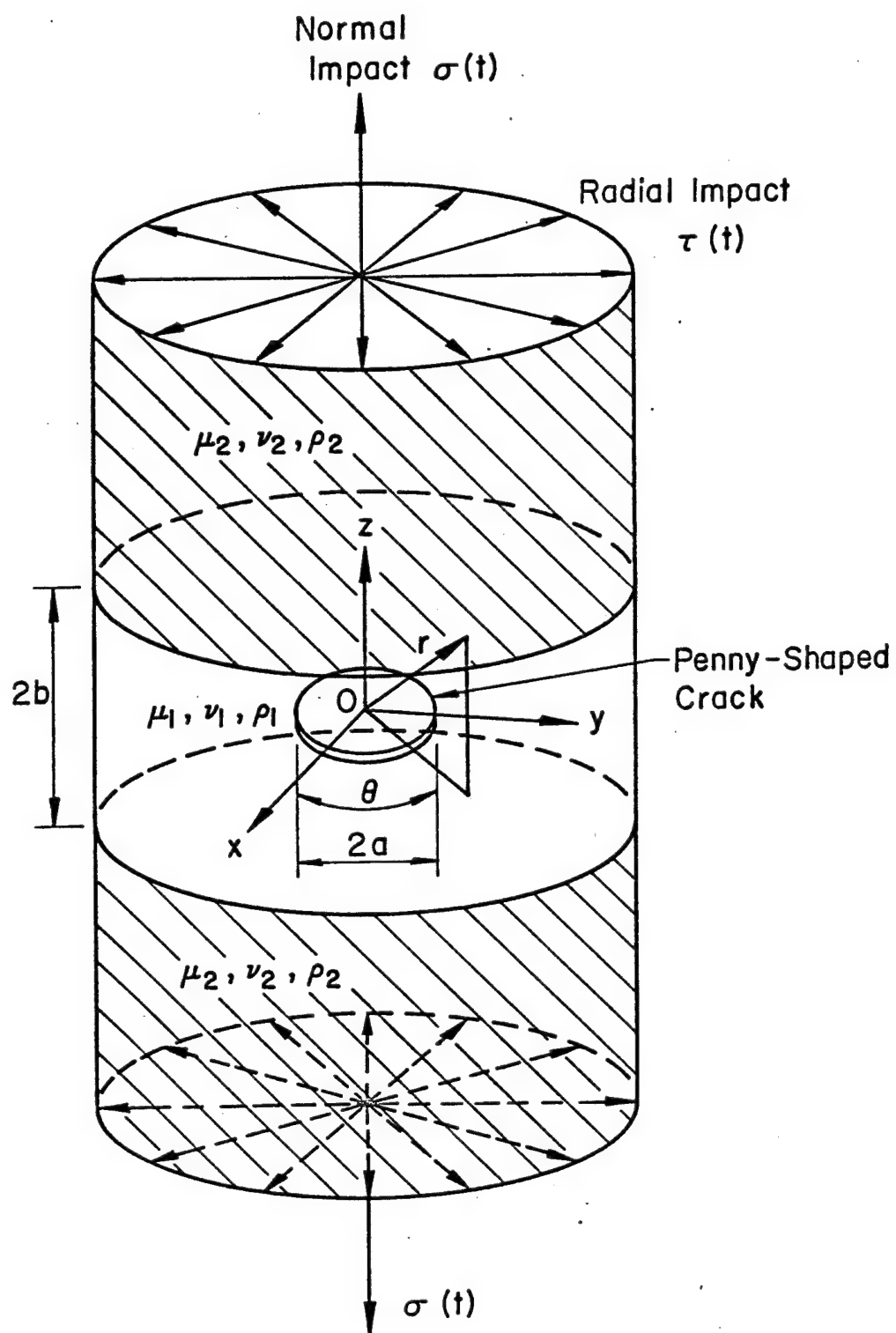


Figure 1 - Penny-shaped crack embedded in a matrix layer under normal and radial impact

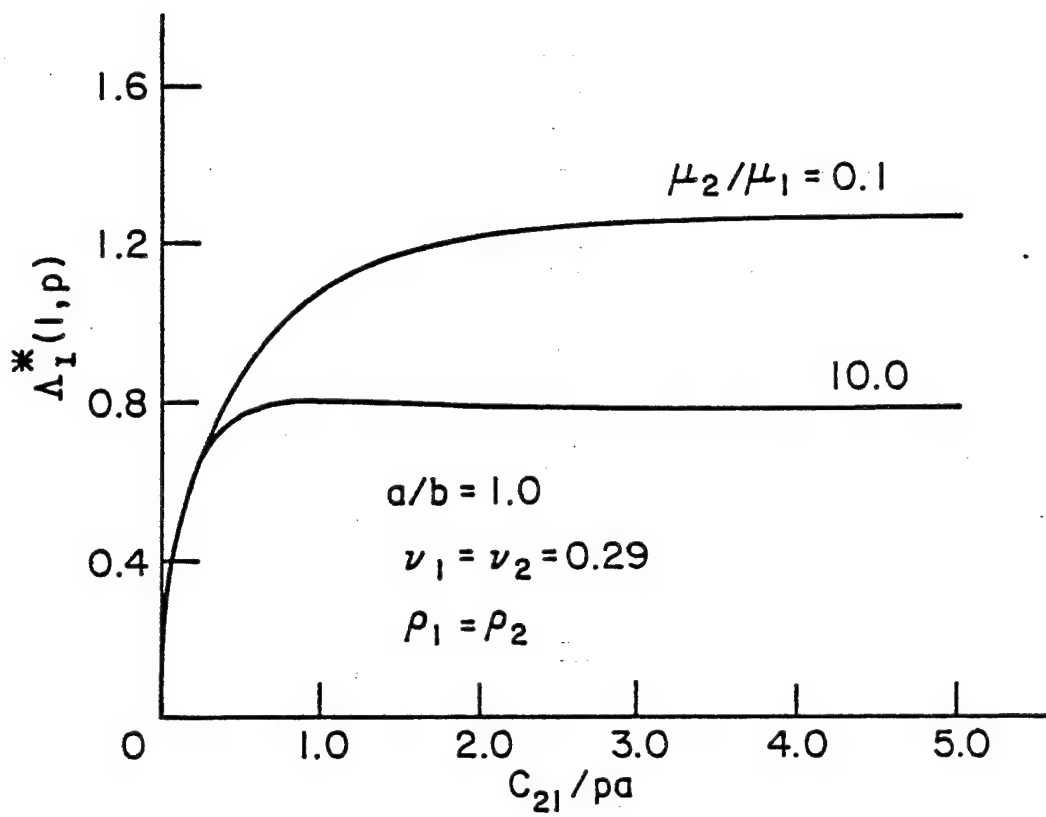


Figure 2 - Plot of $\Delta_I^*(1, p)$ versus c_{21}/pa for $a/b = 1.0$

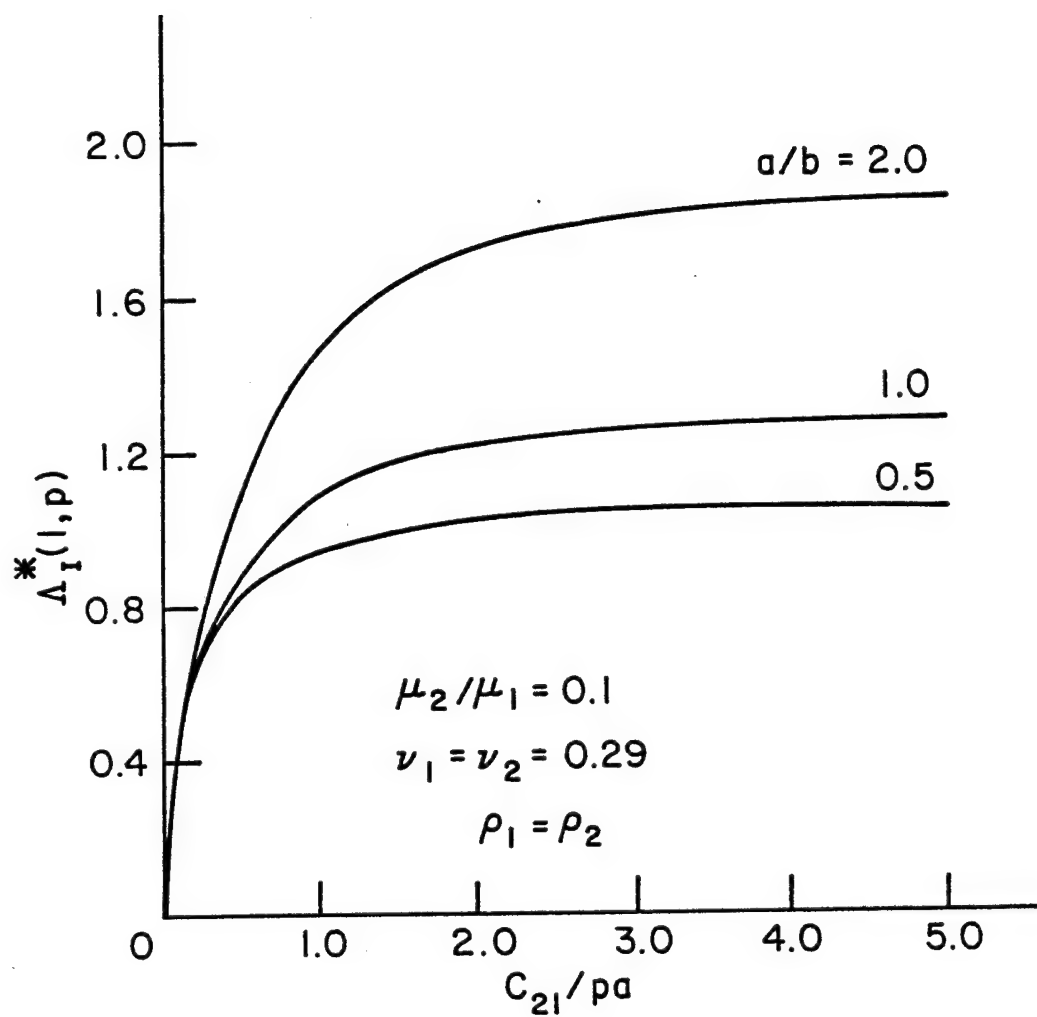


Figure 3 - Plot of $\Delta_I^*(1,p)$ versus c_{21}/pa for $\mu_2/\mu_1 = 0.1$

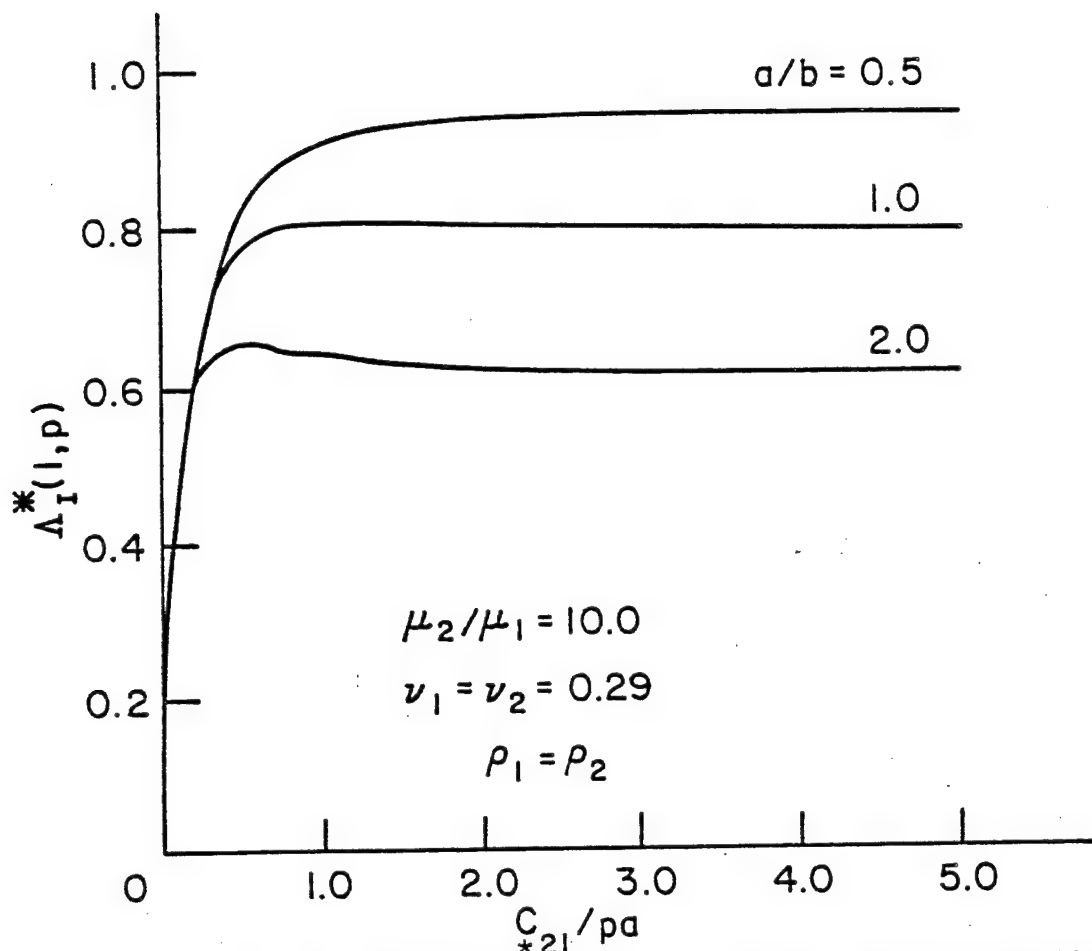


Figure 4 - Plot of $\Delta_I^*(l, p)$ versus c_{2l}/pa for $\mu_2/\mu_1 = 10.0$

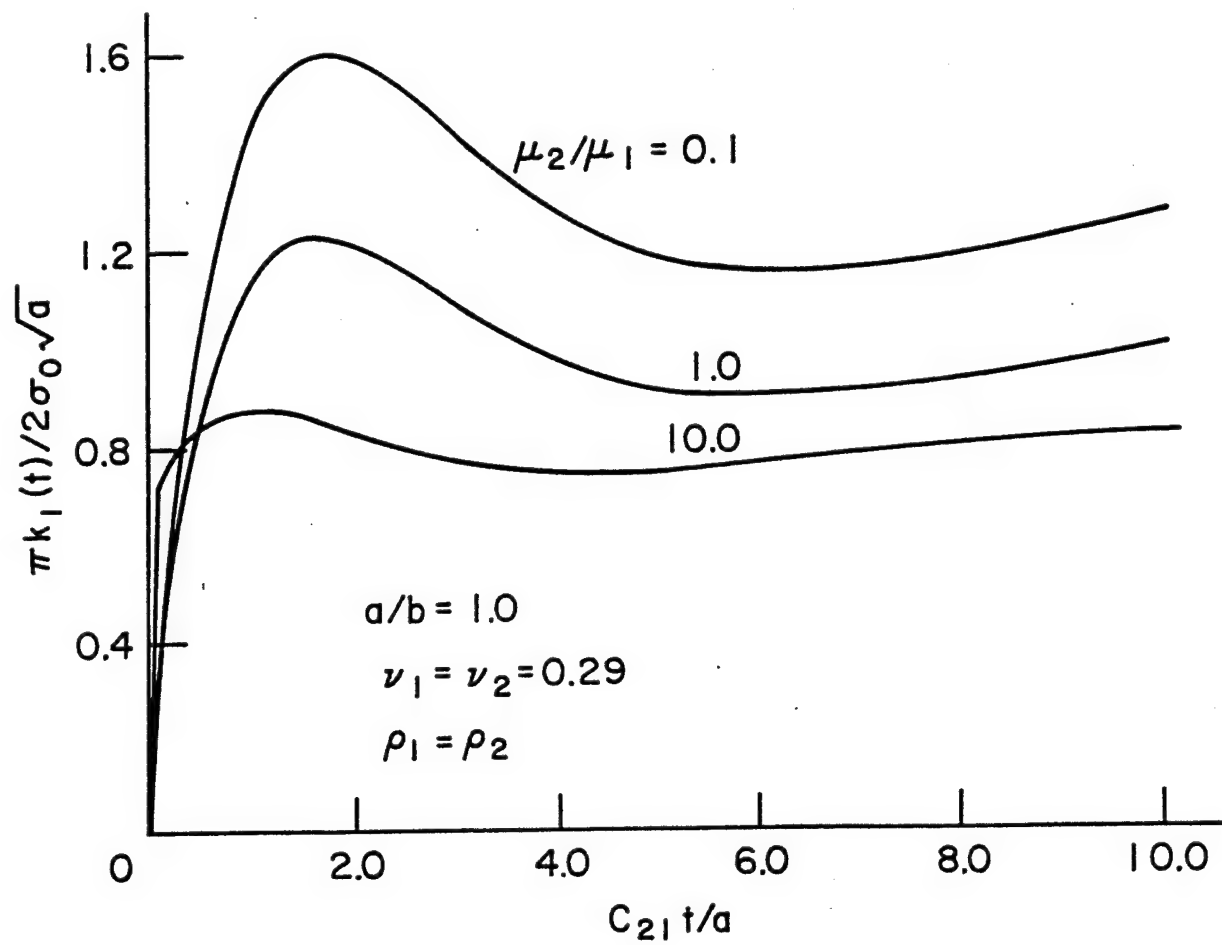


Figure 5 - Dynamic stress intensity factor $k_I(t)$ for penny-shaped crack with $a/b = 1.0$

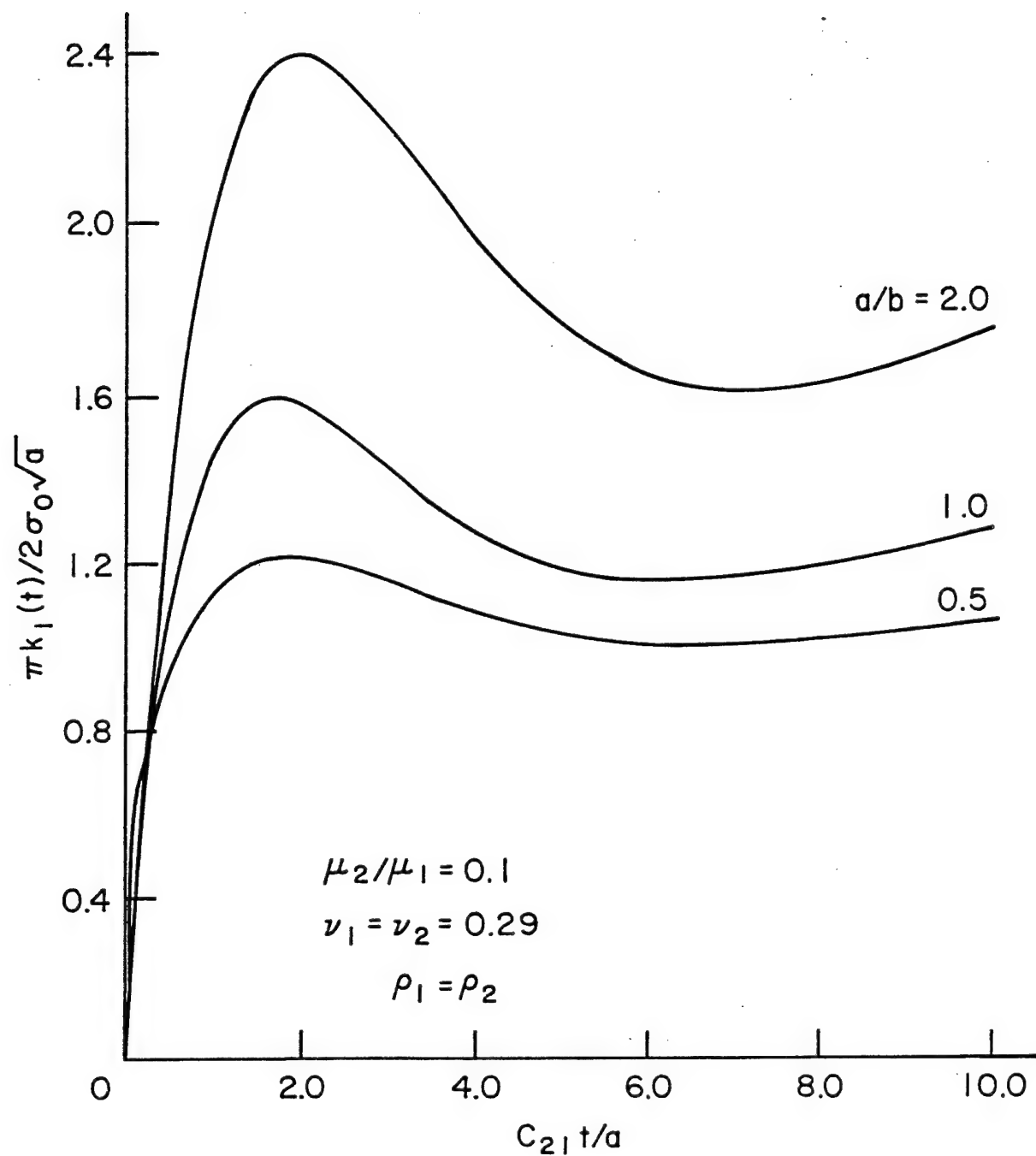


Figure 6 - Dynamic stress intensity factor $k_I(t)$ for penny-shaped crack with $\mu_2/\mu_1 = 0.1$

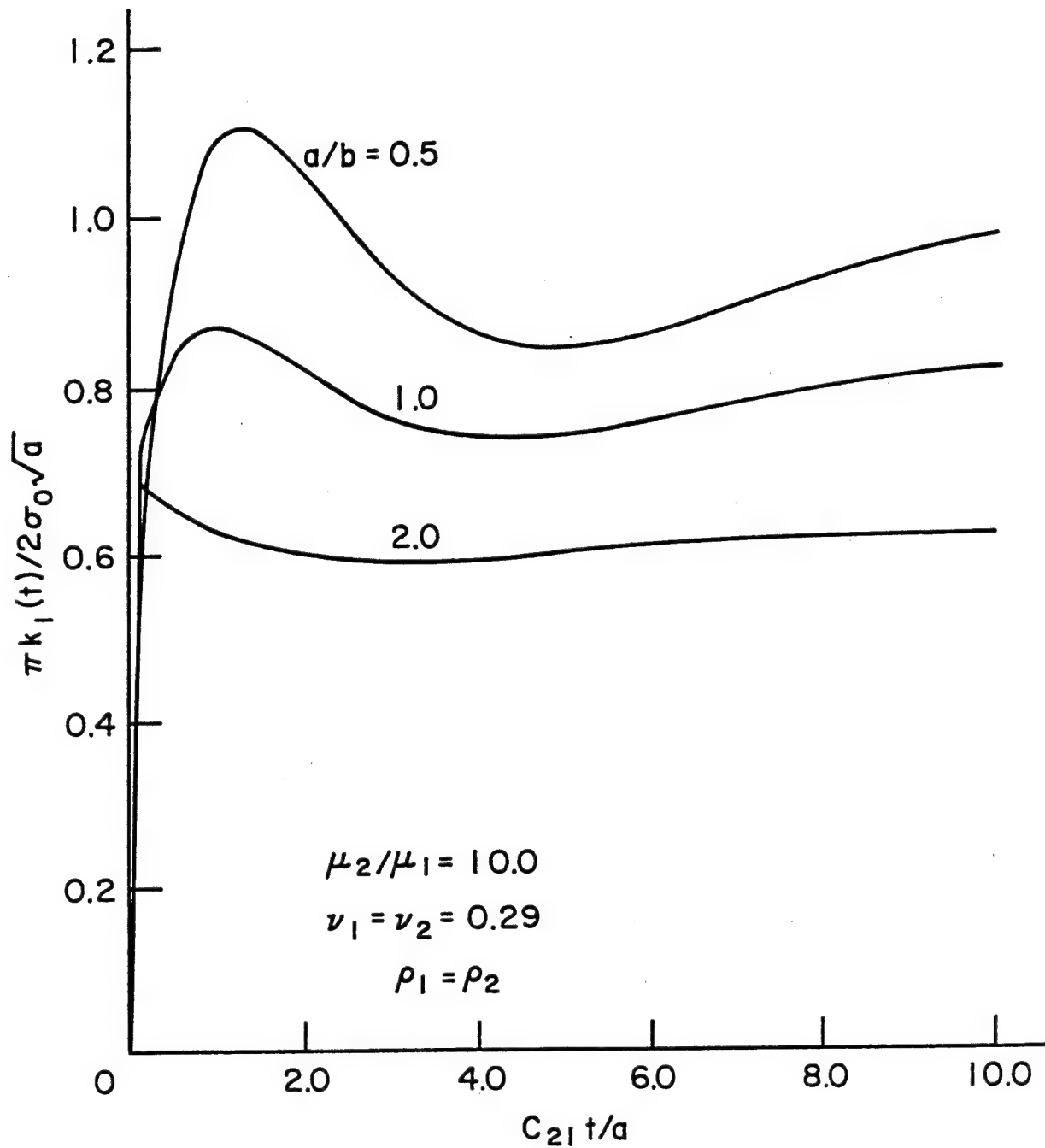


Figure 7 - Dynamic stress intensity factor $k_I(t)$ for penny-shaped crack with $\mu_2/\mu_1 = 10.0$

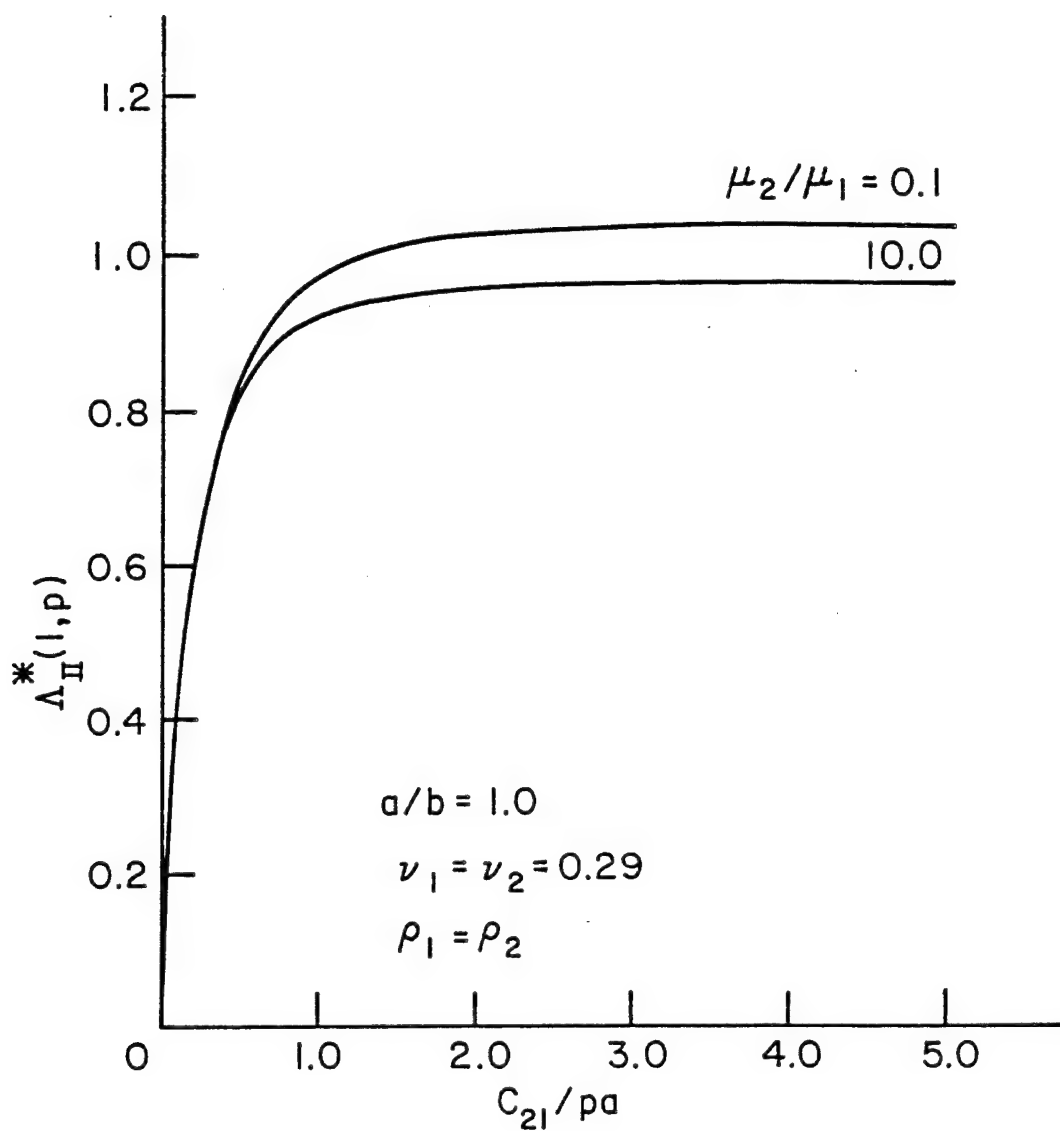


Figure 8 - Variations of $\Lambda_{II}^*(1,p)$ with c_{21}/pa for $a/b = 1.0$

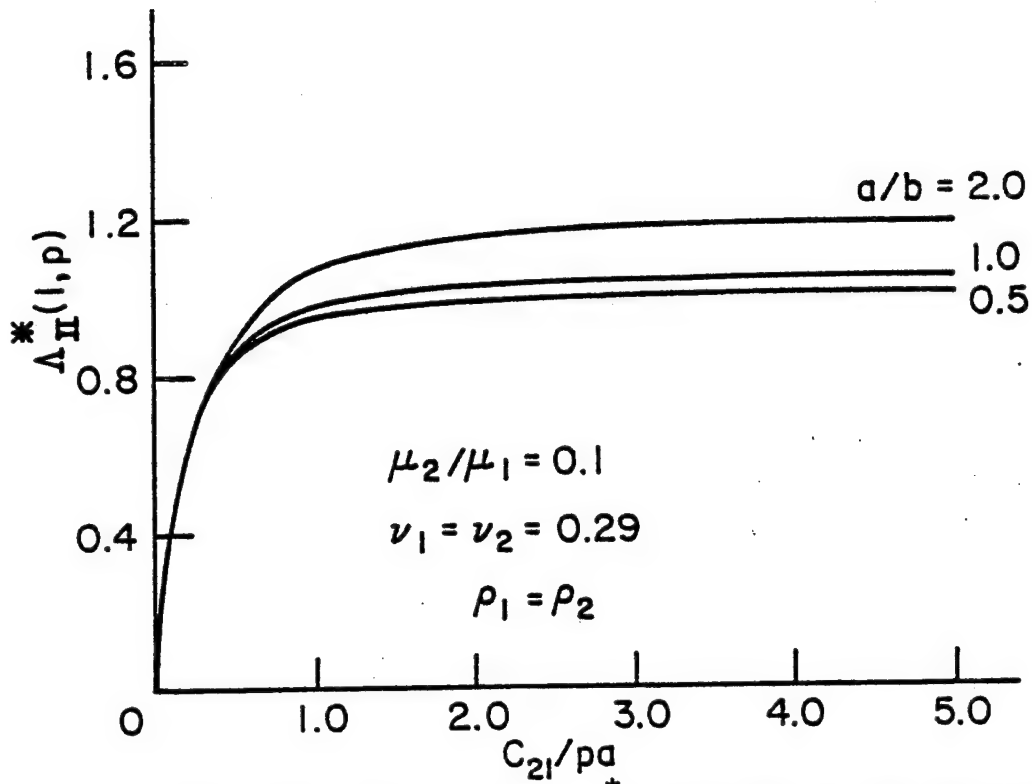


Figure 9 - Variations of $\Lambda_{II}^*(1, p)$ with c_{21}/pa for $\mu_2/\mu_1 = 0.1$ and varying a/b

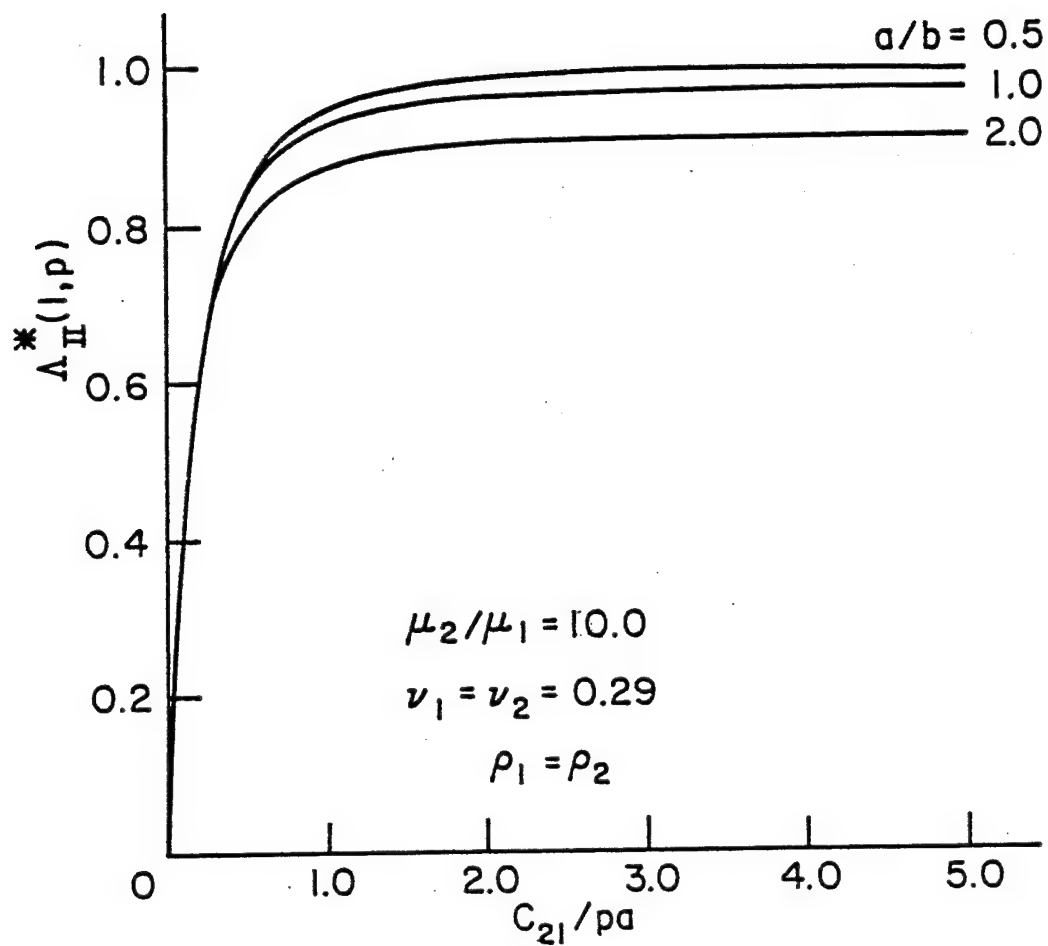


Figure 10 - Variations of $\Lambda_{II}^*(1,p)$ with c_{21}/pa for $\mu_2/\mu_1 = 10$ and varying a/b

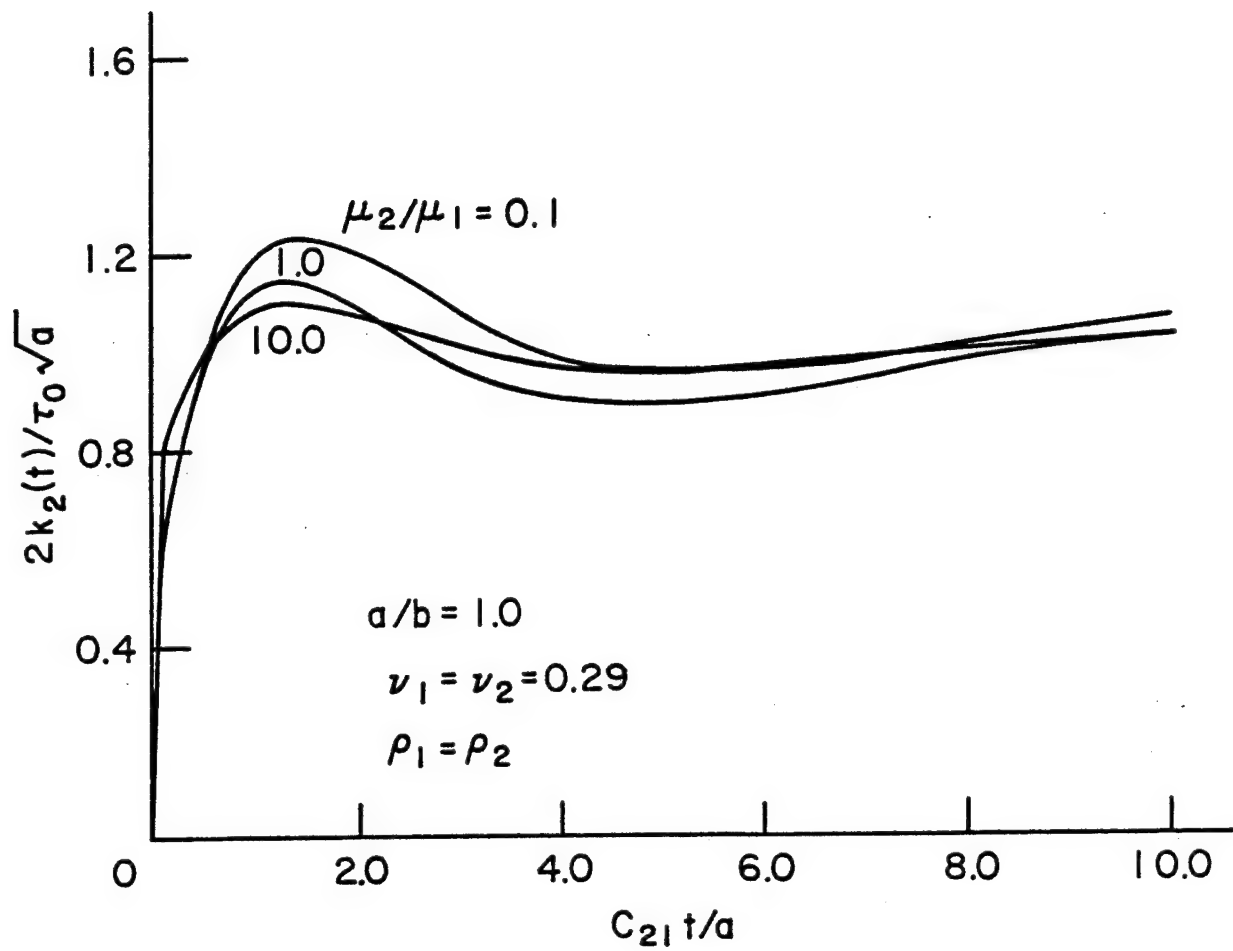


Figure 11 - Stress intensity factor $k_2(t)$ versus time for a penny-shaped crack with $a/b = 1.0$

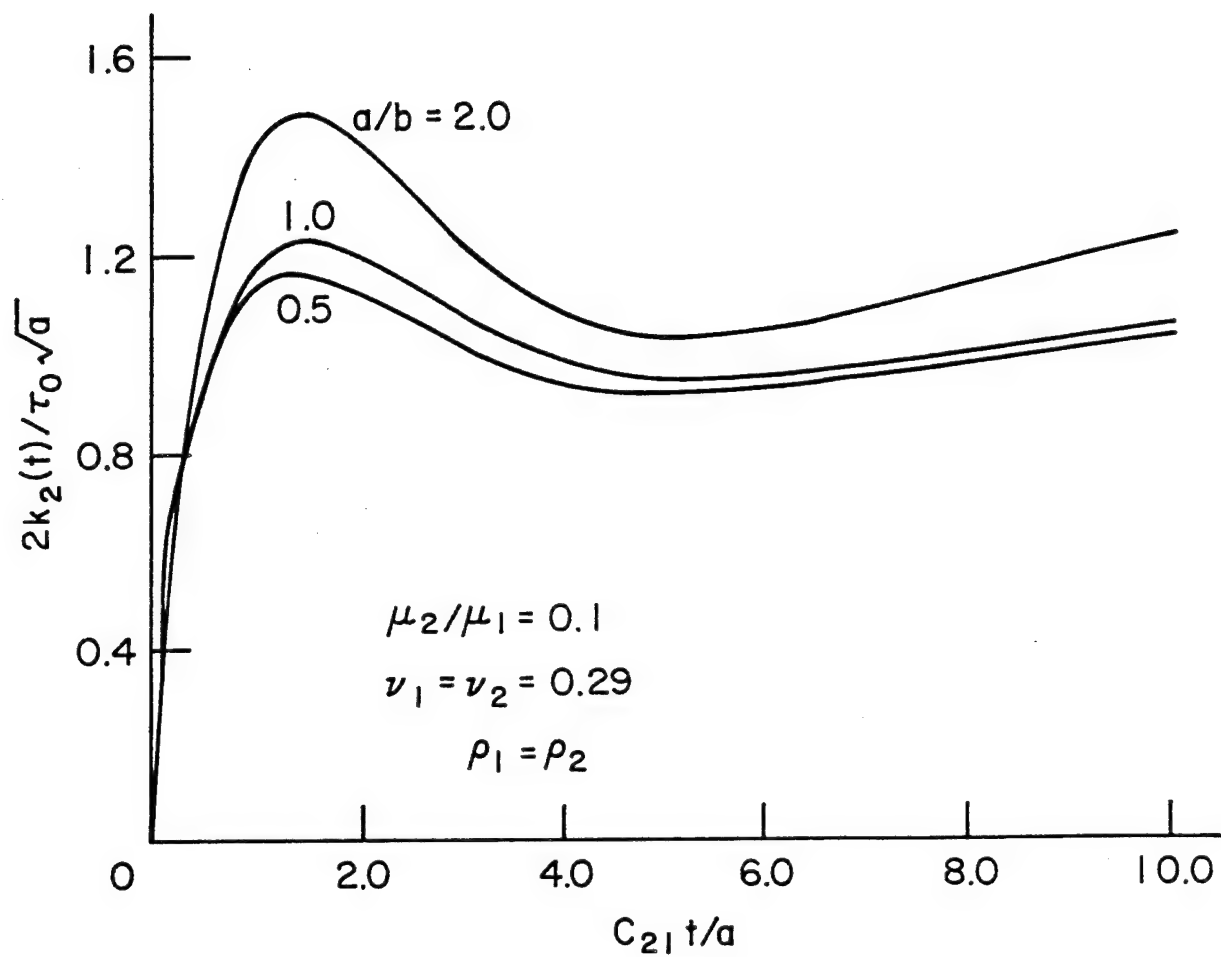


Figure 12 - Stress intensity factor $k_2(t)$ versus time for a penny-shaped crack with $\mu_2/\mu_1 = 0.1$

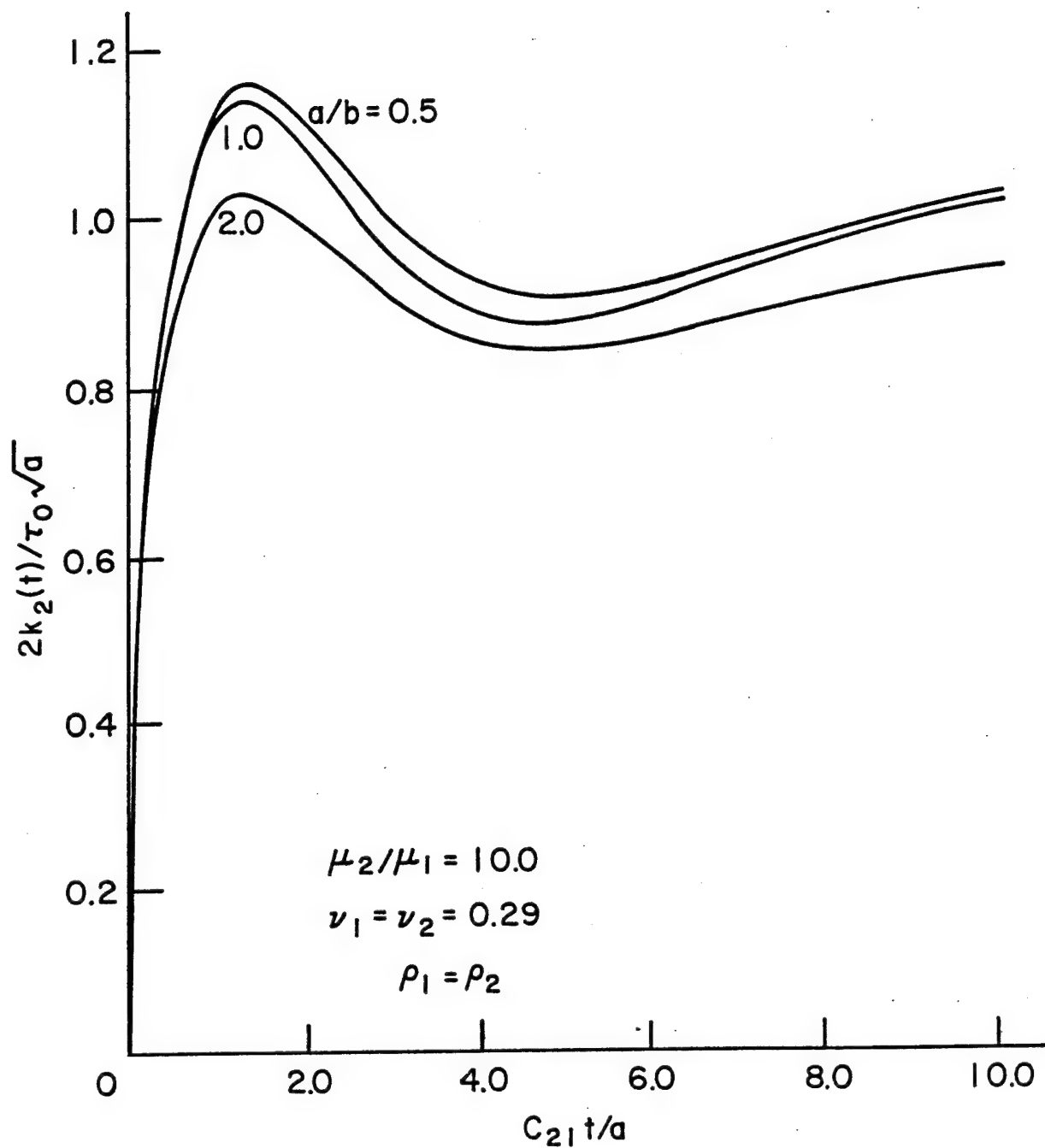


Figure 13 - Stress intensity factor $k_2(t)$ versus time for a penny-shaped crack with $\mu_2/\mu_1 = 10.0$

Axial Impact

```

3      PROGRAM BETA(INPUT,OUTPUT,PUNCH,PLOT,TAPE 99=PLOT)
3      REAL NON(4),F(4,4,1),G(4,4),D(4),PT(4)
3      REAL B(4),C(4)
3      REAL LP(50),DTA(50)
3      EQUIVALENCE (NON,B)
3      COMMON K1,K2,K3,K4
3      COMMON/AUX/H,P,PK1,PK2,BMU,X,Y
3      LP(1)=0.0
4      DTA(1)=0.0
5      READ 2,K1,K2,K3,K4
20     2    FORMAT(I2)
*     K1 = ORDER OF SYSTEM OF EQUATIONS
*     K2 = NO. OF DISTINCT KERNELS
*     K3 = NO. OF DATA POINTS
*     K4 = NO. OF DATA SETS TO BE EVALUATED
*     SET UP DATA POINTS
20     AK=K3
22     DO 5 N=1,K3
23     AN=N
24     5    PT(N)=AN/AK
*     SET UP INTEGRATION MATRIX
31     M=K3-2
33     N=K3-1
34     A=K3
35     A=1./(3.*A)
37     DO 10 K=2,M,2
41     10    D(K)=2.*A
46     DO 15 K=1,N,2
47     15    D(K)=4.*A
54     D(K3)=A
*     CALCULATE NONHOMOGENEOUS TERMS
56     RHS=1.0
57     DO 22 I=1,K2
61     PRINT 9
64     9    FORMAT(1H1)
64     READ 61,BMU
72     61   FORMAT(F10.5)
72     DO 999 II=1,K4
74     DO 35 N=1,K3
75     35   NON(N)=RHS*PT(N)
*     CALCULATE KERNEL MATRICES
102    CALL CONST(I)
103    DO 20 N=1,K3
105    DO 20 M=1,K3
106    IF (M-N)25,30,30
111    25   F(M,N,I)=F(N,M,I)
120    GO TO 20
120    30   F(M,N,I)=FU(I,PT(M),PT(N))
131    20   CONTINUE
136    CALL CHANGE(F,G,D,I)
141    CALL LINEQ(G,B,C,K3)
144    DO 40 L=1,K3
146    PRINT 6,PT(L),NON(L)
155    6    FORMAT(5X,F8.4,F15.6)
155    40   CONTINUE
160    LP(II+1)=NON(K3)
162    DTA(II+1)=P
164    999   CONTINUE
166    PUNCH 66,(DTA(IX),LP(IX),IX=1,19)
202    66   FORMAT(2F10.5)
202    CALL LAPINV(DTA,LP)
204    22   CONTINUE
207    END

```

```

6      FUNCTION SIMP(I,A,B)
6      COMMON/AUX/H,P,PK1,PK2,BMU,X,Y
10     DEL=0.25*(B-A)
12     IF (DEL)40,45,50
13     45   SIMP=0.0
13     RETURN
14     50   CONTINUE
14     SA=Z(I,A)+Z(I,B)
26     SB=Z(I,A+2.*DEL)
35     SC=Z(I,A+DEL)+Z(I,A+3.*DEL)

```

```

53      S1=(DEL/3.)*(SA+2.*SB+4.*SC)
61      IF(S1.EQ.0.0) GO TO 45
62      K=8
63      35  S3=SB+SC
65      DEL=0.5*DEL
67      SC=Z(I,A+DEL)
75      J=K-1
77      DO 5 N=3,J,2
100     AN=N
101     5   SC=SC+Z(I,A+AN*DEL)
113     S2=(DEL/3.)*(SA+2.*SB+4.*SC)
122     DIF=ABS((S2-S1)/S1)
125     ER=0.01
127     IF(DIF-ER) 30,25,25
131     30  SIMP=S2
133     RETURN
133     25  K=2*K
134     S1=S2
136     IF(K-2048) 35,35,40
140     40  PRINT 42,I,A,E
152     42  FORMAT(5X,* INT. DOES NOT CONVERGE *,I3,2F9.4)
152     PRINT 60,X,Y
162     60  FORMAT(2F10.5)
162     DO 70 J=1,10
166     DIP=J
167     DIP=DIP/10.
171     W=Z(I,DIF)
175     PRINT 60,W
202     70  CONTINUE
206     CALL EXIT
207     END

```

```

7      SUBROUTINE CHANGE(F,G,D,I)
7      REAL F(4,4,1),G(4,4),D(4)
7      COMMON K1,K2,K3,K4
10     DO 10 N=1,K3
11     DO 10 M=1,K3
24     10  G(M,N)=F(M,N,I)*D(N)
30     DO 20 N=1,K3
31     20  G(N,N)=G(N,N)+1.0
40     RETURN
41     END

```

```

7      SUBROUTINE LINEQ(A,B,T,N)
7      REAL A(N,N),E(N),T(N)
10     DO 5 I=2,N
17     5   A(I,1)=A(I,1)/A(1,1)
20     DO 10 K=2,N
22     M=K-1
23     DO 15 I=1,N
33     15  T(I)=A(I,K)
34     DO 20 J=1,M
41     A(J,K)=T(J)
43     J1=J+1
44     DO 20 I=J1,N
55     20  T(I)=T(I)-A(I,J)*A(J,K)
61     CONTINUE
65     A(K,K)=T(K)
66     IF(K.EQ.N) GO TO 10
70     M=K+1
71     DO 25 I=M,N
105    25  A(I,K)=T(I)/A(K,K)
110    10  CONTINUE
111    *  BACK SUBSTITUTE
114    DO 30 I=1,N
116    T(I)=B(I)
121    M=I+1
122    IF(M.GT.N) GO TO 30
132    DO 30 J=M,N
136    30  B(J)=B(J)-A(J,I)*T(I)

```

```

37 K=N+1-I
41 B(K)=T(K)/A(K,K)
46 K1=K-1
50 IF(K1.EQ.0) GO TO 35
51 DO 35 J1=1,K1
52 J=K-J1
54 T(J)=T(J)-A(J,K)*B(K)
62 35 CONTINUE
67 RETURN
67 END

```

```

6 FUNCTION FU(I,A,B)
6 COMMON/AUX/H,P,PK1,PK2,BMU,X,Y
7 X=A
7 Y=B
10 IF(A*B)5,10,5
11 FU=0.0
12 RETURN
13 5 SUM=SIMP(I,0.0,5.0)
20 ER=0.01
21 DEL=5.0
23 20 UP=DEL+5.0
25 ADDL=SIMP(I,DEL,UP)
32 DEL=UP
33 TEST=ABS(ADDL/SUM)
36 SUM=SUM+ADDL
37 IF(TEST-ER)15,20,20
41 15 FU=SQRT(X*Y)*SUM
47 RETURN
47 END

```

```

3 SUBROUTINE CONST(I)
3 COMMON/AUX/H,P,PK1,PK2,BMU,X,Y
3 PR1=0.29
6 PR2=0.29
6 PK1=SQRT(((1.-2.*PR1)/(2.*(1.-PR1)))
15 PK2=SQRT(((1.-2.*PR2)/(2.*(1.-PR2)))
24 READ 1,P
31 1 FORMAT(F10.5)
31 HH=0.1
33 HH=10.0
34 HH=5.0
36 HH=4.0
37 HH=1.0
41 HH=0.5
42 HH=2.0
44 H=1./HH
45 PRINT 2,BMU,PR1,PR2,HH,P
62 2 FORMAT(/////5X,* MU2/MU1 =*F6.2,* NU1 =*F4.2,* NU2 =*F4.2/////5X,* A
1/H =*F4.2,* C21/PA =*F4.2)
62 RETURN
63 END

```

```

5 FUNCTION Z(I,S)
5 COMMON/AUX/H,P,PK1,PK2,BMU,X,Y
23 BESJH(A)=SQRT(2.*A/PI)*SIN(A)/A
25 PI=3.1415926
27 IF(S-0.0)5,5,10
30 5 Z=0.0
31 RETURN
31 10 CONTINUE
31 PP=P*P
33 C1=PK1*PK1
34 C2=PK2*PK2
36 CC=1.-C1
40 GA=SQRT(S*S+C1/PP)
46 GB=SQRT(S*S+1./PP)
55 GC=SQRT(S*S+C2/BMU/PP)
64 GD=SQRT(S*S+1./BMU/PP)
73 AA=S*S+1./PP/2.
77 AB=1.-BML

```

```

100 AC=S*S-GC*GD
103 AD=(GB-GD)/AC/PP/2.*BMU
110 AE=(GB+GD)/AC/PP/2.*BMU
115 AF=(S*S-GA*GD)/AC/PP/2.*EMU
123 AG=(S*S+GA*GD)/AC/PP/2.*EMU
131 AH=(S*S-GB*GC)/AC/PP/2.*BMU
137 AI=(S*S+GB*GC)/AC/PP/2.*BMU
145 AJ=(GA-GC)/AC/PP/2.*BMU
152 AK=(GA+GC)/AC/PP/2.*BMU
157 A1=-(AB*GB-AD)
162 A2=AB*GB-AE
164 A3=AA-BMU*S*S-AF
171 A4=AA-BMU*S*S-AG
174 A5=-AA+EMU*S*S-AH
200 A6=-AA+BMU*S*S-AI
203 A7=S*(AE*GA-AJ)
206 A8=-S*(AE*GA-AK)
211 BA=A1*A6-A2*A5
214 BB=A3*A6-S*A2*A7
217 BC=A4*A6-S*A2*A8
222 BD=S*A1*A7-A3*A5
226 BE=S*A1*A8-A4*A5
231 B1=BB/BA
233 B2=BC/BA
235 B3=BD/BA
236 B4=BE/BA
240 EA=2.*GA*H
242 EB=2.*GB*H
244 EC=(EA+EB)/2.
246 ED=2.*EC
247 E1=EXP(-EA)
252 E2=EXP(-EB)
256 E3=EXP(-EC)
262 E4=EXP(-ED)
266 DL=B2+B3*E4+B4*E2+B1*E1
275 D1=2.*PP/CC/GE/DL
301 D2=AA*AA-S*S*GA*GB
307 D3=B2-B3*E4
312 D4=2.*AA*(GB*(B1*B4-B2*B3)-S*S*GA)*E3
321 D5=(AA*AA+S*S*GA*GB)*(B4*E2-B1*E1)
332 F=D1*(D2*D3+D4+D5)
337 Z=(F-S)*EESJH(S*X)*BESJH(S*Y)
347 RETURN
350 END

```

C
C
C

```

SUBROUTINE LAPINV(GLAM,PHI)
THIS PROGRAM EVALUATES THE COEFFICIENTS FOR SERIES
OF JACOBI POLYNOMIALS WHICH REPRESENTS A LAPLACE
INVERSION INTEGRAL
REAL MUL
DIMENSION A(50),GLAM(50),PHI(50),C(4,50)
DIMENSION BK(101),TT(101)
COMMON/2/TT,TF,DT,MN,EK,TT
READ 1,NN,MN,MM
1 FORMAT(3I2)
READ 2,TT,TF,DT
2 FORMAT(3F10.5)
PRINT 99
99 FORMAT(1H1)
CALL SPLICE(GLAM,PHI,MM,C)
PRINT 101
101 FORMAT(/////5X,* GLAM PHI *)
PRINT 102,(GLAM(I),PHI(I),I=1,MM)
102 FORMAT(5X,F10.5,5X,F10.5)
M11=MM-1
PRINT 300
300 FORMAT(/////5X,* C(1,J) C(2,J) C(3,J) C(4
1,J) *)
PRINT 103,((C(I,J),I=1,4),J=1,M11)
103 FORMAT(5X,F10.5,5X,F10.5,5X,F10.5,5X,F10.5)
PRINT 99
DO 10 I=1,NN
READ 3,BET,DEL
3 FORMAT(2F10.5)
PRINT 98,BET,DEL

```

```

140 98 FORMAT(/////5X,*BETA =*F5.3,* DELTA =*F5.3)
140 DO 11 L=1,MN
143 AL=L
144 S=1./(AL+BET)/DEL
150 CALL SPLINE(GLAM,PHI,MM,C,S,G)
153 F=G*S
155 IF (AL-2.) 81,82,83
161 81 A(1)=(1.+BET)*DEL*F
165 GO TO 11
165 82 A(2)=((2.+BET)*DEL*F-A(1))*(3.+BET)
175 GO TO 11
175 83 CONTINUE
175 TOP=1.
177 L1=L-1
201 AL1=L1
202 DO 12 J=1,L1
203 AJ=J
204 TOP=AJ*TOP
206 12 CONTINUE
210 L2=2*L-1
212 BOT=1.
214 DO 13 J=L,L2
215 AJ=J
216 BOT=(AJ+BET)*BOT
221 13 CONTINUE
223 MUL=BOT/TOP
225 SUM=0.0
226 DO 14 N=1,L1
227 AN=N
230 IF (AN-2.) 85,86,87
233 85 TOC=1.
235 GO TO 88
235 86 TOD=AL1
237 GO TO 88
237 87 CONTINUE
237 TOD=1.
241 ICH=L1-(N-2)
244 DO 15 J=ICH,L1
245 AJ=J
246 TOC=AJ*TOD
250 15 CONTINUE
252 88 CONTINUE
252 BOD=1.
254 JA=L1+N
256 DO 16 J=L,JA
260 AJ=J
261 BOD=BOD*(AJ+BET)
264 16 CONTINUE
266 CO=TOD/BOD
270 SUM=SUM+CO*A(N)
273 14 CONTINUE
275 A(L)=MUL*(DEL*F-SUM)
301 11 CONTINUE
304 CALL JACSER(DEL,A,BET)
306 CALL NAMPLT
307 CALL QIKSET(6.0,0.0,0.0,0.0,0.0,0.0,0.0)
313 CALL QIKSAX(3,3)
315 CALL QIKFLT(TT,BK,101)
320 CALL ENDFLT
321 10 CONTINUE
325 999 CONTINUE
325 RETURN
326 END

```

```

6 SUBROUTINE JACSER(D,C,B)
6 DIMENSION C(50),SF(50),P(50)
6 DIMENSION BK(101),TT(101)
6 COMMON/2/TT,TF,DT,MN,BK,TT
7 TT(1)=0.0
7 BK(1)=0.0
10 LM=1
11 T=TT
12 T=T+DT
14 X=2.*EXP(-D*T)-1.
24 CALL JACOBI(MN,X,B,P)

```

```

26      SF(1)=C(1)*P(1)
32      DO 10 L=2,MN
33      L1=L-1
35      AL=L
36      SF(L)=SF(L1)+C(L)*P(L)
43      10 CONTINUE
45      PRINT 97,T,X
55      97 FORMAT(/////5X,* T =*F6.3,* X =*F10.5)
55      PRINT 96
61      96 FORMAT(/////5X,* I C(I) *,5X,* N F(T) *)
61      DO 11 I=1,6
65      PRINT 95,I,C(I),I,SF(I)
105      95 FORMAT(5X,I2,F10.2,5X,I2,F10.5)
105      11 CONTINUE
111      LM=LM+1
113      BK(LM)=SF(5)
115      TT(LM)=T
117      IF(T.LE.TF) GO TO 12
121      RETURN
122      END

```

```

C      SUBROUTINE JACOBI(N,X,B,PB)
C      THIS PROGRAM CALCULATES JACOBI POLYNOMIALS OF ORDER
C      K-1 WITH ARG X AND PARAMETER B GT -1
7      DIMENSION PB(N)
7      AN=N
10      IF (AN-2.) 1,2,3
12      1 PB(1)=1.
14      RETURN
14      2 PB(1)=1.
16      PB(2)=X-B*(1.-X)/2.
21      RETURN
22      3 BSQ=B*B
23      BONE=B+1.
25      PB(1)=1.
26      PB(2)=X-B*(1.-X)/2.
31      DO 4 K=3,N
33      AK=K
34      AK1=AK-1.
36      AK2=AK-2.
40      K1=K-1
42      K2=K-2
43      CO1=((2.*AK1)+B)*X
46      CO1=((2.*AK2)+B)*CO1
51      CO1=((2.*AK2)+BONE)*(CO1-BSQ)
56      CO2=2.*AK2*(AK2+B)*((2.*AK1)+B)
64      CO2=2.*AK1*(AK1+B)*((2.*AK2)+B)
71      4 PB(K)=(CO1*PB(K1)-CO2*PB(K2))/CO
102      RETURN
103      END

```

```

11      SUBROUTINE SPLINE(X,Y,M,C,XINT,YINT)
11      DIMENSION X(50),Y(50),C(4,50)
13      IF (XINT-X(1)) 1,10,11
14      10 YINT=Y(1)
15      RETURN
15      11 CONTINUE
21      IF (X(M)-XINT) 1,12,13
23      12 YINT=Y(M)
23      RETURN
23      13 CONTINUE
25      K=M/2
26      N=M
26      2 CONTINUE
32      IF (X(K)-XINT) 3,14,5
34      14 YINT=Y(K)
35      RETURN
35      3 CONTINUE
41      IF (XINT-X(K+1)) 4,15,7
43      15 YINT=Y(K+1)
43      RETURN
43      4 CONTINUE
43      YINT=(X(K+1)-XINT)*(C(1,K)+(X(K+1)-XINT)**2+C(3,K))

```

```

54      YINT=YINT+(XINT-X(K))*(C(2,K)*(XINT-X(K))**2+C(4,K))
65      RETURN
65      5 CONTINUE
65      IF(X(K-1)-XINT)6,16,17
70      6 K=K-1
72      GO TO 4
72      16 YINT=Y(K-1)
74      RETURN
75      17 N=K
77      K=K/2
100      GO TO 2
100      7 LL=K
102      K=(N+K)/2
103      8 CONTINUE
103      IF(X(K)-XINT)3,14,18
106      18 CONTINUE
106      IF(X(K-1)-XINT)6,16,19
111      19 N=K
113      K=(LL+K)/2
114      GO TO 8
115      1 PRINT 101
121 101 FORMAT(* OUT OF RANGE FOR INTERPOLATION *)
121      STOP
123      END

```

```

7      SUBROUTINE SPLICE(X,Y,M,C)
7      DIMENSION X(50),Y(50),D(50),P(50),E(50),C(4,50)
7      DIMENSION A(50,3),B(50),Z(50)
11      MM=M-1
12      DO 2 K=1,MM
15      D(K)=X(K+1)-X(K)
15      P(K)=D(K)/6.
20      2 E(K)=(Y(K+1)-Y(K))/D(K)
26      DO 3 K=2,MM
27      3 B(K)=E(K)-E(K-1)
34      A(1,2)=-1.-D(1)/D(2)
37      A(1,3)=D(1)/D(2)
41      A(2,3)=P(2)-P(1)*A(1,3)
44      A(2,2)=2.*(P(1)+P(2))-P(1)*A(1,2)
50      A(2,3)=A(2,3)/A(2,2)
51      B(2)=B(2)/A(2,2)
53      DO 4 K=3,MM
54      A(K,2)=2.*(P(K-1)+P(K))-P(K-1)*A(K-1,3)
61      B(K)=B(K)-P(K-1)*B(K-1)
65      A(K,3)=P(K)/A(K,2)
70      4 B(K)=B(K)/A(K,2)
74      Q=D(M-2)/D(M-1)
76      A(M,1)=1.+Q+A(M-2,3)
101      A(M,2)=-Q-A(M,1)*A(M-1,3)
105      B(M)=B(M-2)-A(M,1)*B(M-1)
112      Z(M)=B(M)/A(M,2)
114      MN=M-2
116      DO 6 I=1,MN
117      K=M-I
120      6 Z(K)=B(K)-A(K,3)*Z(K+1)
127      Z(1)=-A(1,2)*Z(2)-A(1,3)*Z(3)
133      DO 7 K=1,MM
135      Q=1./(6.*D(K))
140      C(1,K)=Z(K)*Q
143      C(2,K)=Z(K+1)*Q
146      C(3,K)=Y(K)/D(K)-Z(K)*P(K)
154      7 C(4,K)=Y(K+1)/D(K)-Z(K+1)*P(K)
165      RETURN
165      END

```

Torsional impact

```

3      PROGRAM BETA(INPUT,OUTPUT,PUNCH,PLOT,TAPE 99=PLOT)
3      REAL NON(4),F(4,4,1),G(4,4),D(4),PT(4)
3      REAL B(4),C(4)
3      REAL LP(50),DTA(50)
3      EQUIVALENCE (NON,B)
3      COMMON K1,K2,K3,K4
3      COMMON/AUX/H,P,PK1,PK2,BMU,X,Y
3      LP(1)=0.0
4      DTA(1)=0.0
5      READ 2,K1,K2,K3,K4
20     FORMAT(I2)
*      2  K1 = ORDER OF SYSTEM OF EQUATIONS
*      *  K2 = NO. OF DISTINCT KERNELS
*      *  K3 = NO. OF DATA POINTS
*      *  K4 = NO. OF DATA SETS TO BE EVALUATED
*      SET UP DATA POINTS
20     AK=K3
22     DO 5 N=1,K3
23     AN=N
24     5  PT(N)=AN/AK
*      SET UP INTEGRATION MATRIX
31     M=K3-2
33     N=K3-1
34     A=K3
35     A=1./(3.*A)
37     DO 10 K=2,M,2
41     10  D(K)=2.*A
46     DO 15 K=1,N,2
47     15  D(K)=4.*A
54     D(K3)=A
*      CALCULATE NONHOMOGENEOUS TERMS
56     RHS=1.0
57     DO 22 I=1,K2
61     PRINT 9
64     9  FORMAT(1H1)
64     READ 61,BMU
72     61 FORMAT(F10.5)
72     DO 999 II=1,K4
74     DC 35 N=1,K3
75     35  NON(N)=RHS*PT(N)*PT(N)
*      CALCULATE KERNEL MATRICES
102    CALL CONST(I)
104    DO 20 N=1,K3
106    DO 20 M=1,K3
107    IF(M-N)25,30,30
112    25  F(M,N,I)=F(N,M,I)
121    GC TO 20
121    30  F(M,N,I)=FU(I,PT(M),PT(N))
132    20  CONTINUE
137    CALL CHANGE(F,G,D,I)
142    CALL LINEQ(G,B,C,K3)
145    DO 40 L=1,K3
147    PRINT 6,PT(L),NON(L)
156    6  FORMAT(5X,F8.4,F15.6)
156    40  CONTINUE
161    LP(II+1)=NON(K3)
163    DTA(II+1)=P
165    CONTINUE
167    999  PUNCH 66,(DTA(IX),LP(IX),IX=1,19)
203    66  FORMAT(2F10.5)
205    CALL LAPINV(DTA,LP)
210    22  CONTINUE
210    END

```

```

6      FUNCTION SIMP(I,A,B)
6      COMMON/AUX/H,P,PK1,PK2,BMU,X,Y
10     DEL=0.25*(B-A)
12     IF(DEL)40,45,50
13     SIMP=0.0
14     RETURN
14     50  CONTINUE
14     SA=Z(I,A)+Z(I,B)
16     SB=Z(I,A+2.*DEL)
15     SC=Z(I,A+DEL)+Z(I,A+3.*DEL)

```

```

53      S1=(DEL/3.)*(SA+2.*SB+4.*SC)
61      IF(S1.EQ.0.0) GO TO 45
62      K=8
63      35  SB=SB+SC
65      DEL=0.5*DEL
67      SC=Z(I,A+DEL)
75      J=K-1
77      DO 5 N=3,J,2
100     AN=N
101     5   S(=SC+Z(I,A+AN*DEL)
113     S2=(DEL/3.)*(SA+2.*SB+4.*SC)
122     DIF=ABS((S2-S1)/S1)
125     ER=0.01
127     IF(DIF-ER)30,25,25
131     30  SIMP=S2
133     RETURN
133     25  K=2*K
134     S1=S2
136     IF(K-2048)35,35,40
140     40  PRINT 42,I,A,B
152     42  FORMAT(5X,* INT. DOES NOT CONVERGE *,I3,2F9.4)
152     PRINT 60,X,Y
162     60  FORMAT(2F10.5)
162     DO 70 J=1,10
166     DIP=J
167     DIP=DIP/10.
171     W=Z(I,DIP)
175     PRINT 60,W
202     70  CONTINUE
206     CALL EXIT
207     END

```

```

7      SUBROUTINE CHANGE(F,G,D,I)
7      REAL F(4,4,1),G(4,4),D(4)
7      COMMON K1,K2,K3,K4
10     DO 10 N=1,K3
11     DC 10 M=1,K3
11     G(M,N)=F(M,N,I)*D(N)
24     10  CONTINUE
30     DO 20 N=1,K3
31     20  G(N,N)=G(N,N)+1.0
40     RETURN
41     END

```

```

7      SUBROUTINE LINEQ(A,B,T,N)
7      REAL A(N,N),B(N),T(N)
10     DO 5 I=2,N
17     5   A(I,1)=A(I,1)/A(1,1)
20     DO 10 K=2,N
22     M=K-1
23     DO 15 I=1,N
33     15  T(I)=A(I,K)
34     DC 20 J=1,M
41     A(J,K)=T(J)
43     J1=J+1
44     DC 20 I=J1,N
55     20  T(I)=T(I)-A(I,J)*A(J,K)
61     CONTINUE
65     A(K,K)=T(K)
66     IF(K.EQ.N) GO TO 10
70     M=K+1
71     DO 25 I=M,N
105     25  A(I,K)=T(I)/A(K,K)
110     10  CONTINUE
110     *  BACK SUBSTITUTE
110     DO 30 I=1,N
111     T(I)=B(I)
114     M=I+1
116     IF(M.GT.N) GO TO 30
117     DO 30 J=M,N
118     B(J)=B(J)-A(J,I)*T(I)
122     30  CONTINUE
126     DO 35 I=1,N

```

```

7 K=N+1-I
41 B(K)=T(K)/A(K,K)
46 K1=K-1
50 IF(K1.EQ.0) GO TO 35
51 DO 35 J1=1,K1
52 J=K-J1
54 T(J)=T(J)-A(J,K)*B(K)
62 35 CONTINUE
67 RETURN
67 END

```

```

6 FUNCTION FU(I,A,B)
6 COMMON/AUX/H,P,PK1,PK2,BMU,X,Y
7 X=A
7 Y=B
10 IF(A*B)5,10,5
11 FU=0.0
12 RETURN
13 5 SUM=SIMP(I,0.0,5.0)
14 ER=0.01
15 DEL=5.0
20 UP=DEL+5.0
21 ADDL=SIMP(I,DEL,UP)
22 DEL=UP
23 TEST=ABS(ADDL/SUM)
24 SUM=SUM+ADDL
25 IF(TEST-ER)15,20,20
15 FU=SQRT(X*Y)*SUM
17 RETURN
17 END

```

```

3 SUBROUTINE CONST(I)
3 COMMON/AUX/H,P,PK1,PK2,BMU,X,Y
3 PR1=0.29
3 PR2=0.29
6 PK1=SQRT((1.-2.*PR1)/(2.*(1.-PR1)))
6 PK2=SQRT((1.-2.*PR2)/(2.*(1.-PR2)))
15 READ 1,P
24 1 FORMAT(F10.5)
31 H=5.0
31 HH=0.2
33 HH=0.5
34 HH=1.0
36 H=2.0
37 H=1./HH
41 PRINT 2,BMU,PR1,PR2,HH,P
42 2 FORMAT(//////5X,* MU2/MU1 =*F6.2,* NU1 =*F4.2,* NU2 =*F4.2///5X,* A
57 1/H =*F4.2,* C21/PA =*F4.2)
57 RETURN
60 END

```

```

5 FUNCTION Z(I,S)
5 COMMON/AUX/H,P,PK1,PK2,BMU,X,Y
26 BESJT(A)=SQRT(2.*A/PI)*(SIN(A)/A-A-COS(A)/A)
30 PI=3.1415926
32 IF(S-0.0)5,5,10
33 5 Z=0.0
34 RETURN
36 10 C N T I N U E
37 PP=P*P
38 GB=SQRT(S*S+1./PP)
39 GD=SQRT(S*S+1./BMU/PP)
45 AA=1.-BMU*GD/GB
54 AB=1.+BMU*GD/GB
60 AC=1.-AA/AB*EXP(-2.*GB*H)
62 AD=1.+AA/AB*EXP(-2.*GB*H)
72 F=GB*AC/AD
80 Z=(F-S)*BESJT(S*X)*BESJT(S*Y)
85 RETURN
16 END
17

```

```

C      SUBROUTINE LAPINV (GLAM, PHI)
C      THIS PROGRAM EVALUATES THE COEFFICIENTS FOR SERIES
C      OF JACOBI POLYNOMIALS WHICH REPRESENTS A LAPLACE
      INVERSION INTEGRAL
      REAL MUL
      DIMENSION A(50), GLAM(50), PHI(50), C(4, 50)
      DIMENSION BK(101), TT(101)
      COMMON/2/TI, TF, DT, MN, BK, TT
      READ 1, NN, MN, MM
      1  FORMAT(3I2)
      READ 2, TI, TF, DT
      2  FORMAT(3F10.5)
      PRINT 99
      99  FORMAT(1H1)
      CALL SPLICE (GLAM, PHI, MM, C)
      PRINT 101
      101  FORMAT(/////5X, *      GLAM      PHI      *)
      PRINT 102, (GLAM(I), PHI(I), I=1, MM)
      102  FORMAT(5X, F10.5, 5X, F10.5)
      M11=MM-1
      PRINT 300
      300  FORMAT(/////5X, *      C(1, J)      C(2, J)      C(3, J)      C(4
      1, J) *)
      PRINT 103, ((C(I, J), I=1, 4), J=1, M11)
      103  FORMAT(5X, F10.5, 5X, F10.5, 5X, F10.5, 5X, F10.5)
      PRINT 99
      DO 10 I=1, NN
      READ 3, BET, DEL
      3  FORMAT(2F10.5)
      PRINT 98, BET, DEL
      98  FORMAT(/////5X, *BETA =*F5.3, * DELTA =*F5.3)
      DO 11 L=1, MN
      AL=L
      S=1./(AL+BET)/DEL
      CALL SPLINE (GLAM, PHI, MM, C, S, G)
      F=G*S
      IF (AL-2.) 81, 82, 83
      81  A(1)=(1.+BET)*DEL*F
      GO TO 11
      82  A(2)=((2.+BET)*DEL*F-A(1))*(3.+BET)
      GO TO 11
      83  CONTINUE
      TOP=1.
      L1=L-1
      AL1=L1
      DO 12 J=1, L1
      AJ=J
      TOP=AJ*TOP
      CONTINUE
      12  L2=2*L-1
      BOT=1.
      DO 13 J=L, L2
      AJ=J
      BOT=(AJ+BET)*BOT
      13  CONTINUE
      MUL=BOT/TOP
      SUM=0.0
      DO 14 N=1, L1
      AN=N
      IF (AN-2.) 85, 86, 87
      85  TOD=1.
      GO TO 88
      86  TOD=AL1
      GO TO 88
      87  CONTINUE
      TOD=1.
      ICH=L1-(N-2)
      DO 15 J=ICH, L1
      AJ=J
      TOD=AJ*TOD
      15  CONTINUE
      88  CONTINUE
      BOO=1.
      JA=L1+N
      DO 16 J=L, JA
      AJ=J

```

```

2 61      BOD=BOD*(AJ+BET)
2 64      16 CONTINUE
266      CO=TOD/BOD
270      SUM=SUM+CO*A(N)
273      14 CONTINUE
275      A(L)=MUL*(DEL*F-SUM)
301      11 CONTINUE
304      CALL JACSER(DEL,A,BET)
306      CALL NAMPLT
307      CALL QIKSET(6.0,0.0,0.0,6.0,0.0,0.0)
313      CALL QIKSAX(3,3)
315      CALL QIKPLT(TT,BK,101)
320      CALL ENDPLT
321      10 CONTINUE
325      999 CONTINUE
325      RETURN
3 26      END

```

```

6      SUBROUTINE JACSER(D,C,B)
6      DIMENSION C(50),SF(50),P(50)
6      DIMENSION BK(101),TT(101)
6      COMMON/2/TT,TF,DT,MN,BK,TT
7      TT(1)=0.0
7      BK(1)=0.0
10     LM=1
11     T=TI
12     12 T=T+DT
14     X=2.*EXP(-D*T)-1.
24     CALL JACOBI(MN,X,B,P)
26     SF(1)=C(1)*P(1)
32     DO 10 L=2,MN
33     L1=L-1
35     AL=L
36     SF(L)=SF(L1)+C(L)*P(L)
43     10 CONTINUE
45     PRINT 97,T,X
55     97 FORMAT(///5X,* T =*F6.3,* X =*F10.5)
55     PRINT 96
61     96 FORMAT(///5X,* I C(I) *,5X,* N F(T) *)
61     DO 11 I=1,6
65     PRINT 95,I,C(I),I,SF(I)
105    95 FORMAT(5X,I2,F10.2,5X,I2,F10.5)
105    11 CONTINUE
111    LM=LM+1
113    BK(LM)=SF(5)
115    TT(LM)=T
117    IF(T.LE.TF) GO TO 12
1 21    RETURN
1 22    END

```

```

C      SUBROUTINE JACOBI(N,X,B,PB)
C      THIS PROGRAM CALCULATES JACOBI POLYNOMIALS OF ORDER
C      K-1 WITH ARG X AND PARAMETER B GT -1
C      DIMENSION PB(N)
C      A=N
C      IF(AN-2.)1,2,3
C      1 PB(1)=1.
C      RETURN
C      2 PB(1)=1.
C      PB(2)=X-B*(1.-X)/2.
C      RETURN
C      3 BSQ=B*B
C      BONE=B+1.
C      PB(1)=1.
C      PB(2)=X-B*(1.-X)/2.
C      DO 4 K=3,N
C      AK=K
C      AK1=AK-1.
C      AK2=AK-2.
C      K1=K-1
C      K2=K-2
C      C01=((2.*AK1)+B)*X
C      C01=((2.*AK2)+B)*C01

```

```

51      CO1=((2.*AK2)+BONE)*(CO1-BSQ)
56      CO2=2.*AK2*(AK2+B)*((2.*AK1)+B)
64      CO=2.*AK1*(AK1+B)*((2.*AK2)+B)
71      4 PB(K)=(CO1*PB(K1)-CO2*PB(K2))/CO
102     RETURN
103     END

```

```

11      SUBROUTINE SPLINE(X,Y,M,C,XINT,YINT)
11      DIMENSION X(50),Y(50),C(4,50)
13      IF(XINT-X(1))1,10,11
14      YINT=Y(1)
15      RETURN
16      11 CONTINUE
17      IF(X(M)-XINT)1,12,13
18      YINT=Y(M)
19      RETURN
20      13 CONTINUE
21      K=M/2
22      N=M
23      2 CONTINUE
24      IF(X(K)-XINT)3,14,5
25      YINT=Y(K)
26      RETURN
27      3 CONTINUE
28      IF(XINT-X(K+1))4,15,7
29      YINT=Y(K+1)
30      RETURN
31      4 CONTINUE
32      YINT=(X(K+1)-XINT)*(C(1,K)*(X(K+1)-XINT)**2+C(3,K))
33      YINT=YINT+(XINT-X(K))*(C(2,K)*(XINT-X(K))**2+C(4,K))
34      RETURN
35      5 CONTINUE
36      IF(X(K-1)-XINT)6,16,17
37      K=K-1
38      GO TO 4
39      16 YINT=Y(K-1)
40      RETURN
41      17 N=K
42      K=K/2
43      GO TO 2
44      7 LL=K
45      K=(N+K)/2
46      8 CONTINUE
47      IF(X(K)-XINT)3,14,18
48      CONTINUE
49      IF(X(K-1)-XINT)6,16,19
50      N=K
51      K=(LL+K)/2
52      GO TO 8
53      1 PRINT 101
54      101 FORMAT(* OUT OF RANGE FOR INTERPOLATION *)
55      STOP
56      END

```

```

7      SUBROUTINE SPLICE(X,Y,M,C)
7      DIMENSION X(50),Y(50),D(50),P(50),E(50),C(4,50)
7      DIMENSION A(50,3),B(50),Z(50)
11      MM=M-1
12      DO 2 K=1,MM
13      D(K)=X(K+1)-X(K)
14      P(K)=D(K)/6.
15      2 E(K)=(Y(K+1)-Y(K))/D(K)
16      DO 3 K=2,MM
17      B(K)=E(K)-E(K-1)
18      A(1,2)=-1.-D(1)/D(2)
19      A(1,3)=D(1)/D(2)
20      A(2,3)=P(2)-P(1)*A(1,3)
21      A(2,2)=2.*(P(1)+P(2))-P(1)*A(1,2)
22      A(2,3)=A(2,3)/A(2,2)
23      B(2)=B(2)/A(2,2)
24      DO 4 K=3,MM
25      A(K,2)=2.*(P(K-1)+P(K))-P(K-1)*A(K-1,3)
26      B(K)=B(K)-P(K-1)*B(K-1)

```

```

65      A(K,3)=P(K)/A(K,2)
70      B(K)=B(K)/A(K,2)
74      Q=D(M-2)/D(M-1)
76      A(M,1)=1.+Q+A(M-2,3)
101     A(M,2)=-Q-A(M,1)*A(M-1,3)
105     B(M)=B(M-2)-A(M,1)*B(M-1)
112     Z(M)=B(M)/A(M,2)
114     MN=M-2
116     DO 6 I=1,MN
117       K=M-I
120     6  Z(K)=B(K)-A(K,3)*Z(K+1)
127     Z(1)=-A(1,2)*Z(2)-A(1,3)*Z(3)
133     DO 7 K=1,MN
135       Q=1./(6.*D(K))
140       C(1,K)=Z(K)*Q
143       C(2,K)=Z(K+1)*Q
146       C(3,K)=Y(K)/D(K)-Z(K)*P(K)
154     7  C(4,K)=Y(K+1)/D(K)-Z(K+1)*P(K)
165     RETURN
165     END

```

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